Communication Enabled Multi-Agent Cooperative Reinforcement Learning Using the Common Information Approach

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Abstract

In this paper using the example of a two-agent matching problem we highlight the applicability of the common information approach from decentralized team problems, for developing low-regret learning schemes for multi-agent cooperative reinforcement learning problems. In our model, one agent is better informed about the environment when compared to the other, and strategically timed (costly) communications from the informed agent are used to increase the common information and to obtain optimal policies when the model is known. We then show how the understanding of the modified information structure leads to a hierarchical bandit formulation when the model is unknown, and where a modified Upper Confidence Bound algorithm achieves close to optimal regret. Finally, we also show how Q-learning based schemes apply to a generalization of the basic model.

Keywords: Multi-agent reinforcement learning, Decentralized team problem, Decentralized POMDP, Common information, Hierarchical bandit.

1. Introduction

The applications of decentralized decision making possibly with uncertainties, are ubiquitous nowadays, including telecommunication/sensor networks, playing strategic games, and autonomous driving, just to name a few (Naghizadeh et al., 2019; K. Zhang, 2019). When the environment is common knowledge, the theory of decentralized stochastic control provides structural results of an optimal policy both analytically and computationally (Nayyar et al., 2013b; Tavafoghi et al., 2018). On the other hand, when the environment is not fully known to the decision makers, they need to learn and explore the environment while trying to maximize their rewards, which gives rise to the problem of multi-agent reinforcement learning (MARL).

Most RL literature assume the underlying model is Markovian. In the single-agent case, these include the model of Markov Decision Process (MDP), and when the agent only has noisy observations on the states, the model of Partially Observable MDP (POMDP). Parallel to the two models in the multi-agent case, decentralized MDP (Dec-MDP) (similar to a stochastic game (SG)) and decentralized POMDP (Dec-POMDP) (similar to a Partially Observable SG (POSG) (Hansen et al., 2004)) have been proposed; among them, Dec-POMDP is the most general one (Bernstein et al., 2002). Depending on the agents' objectives, the models range from fully cooperative, where the agents share a common goal, to fully competitive, where the agents' objectives either sum up to zero, such as the game of Go (K. Zhang, 2019), or can be more complex (Ouyang et al., 2017).

The fully cooperative case, which we will consider in this paper, is well-solved by the *common information approach* when the model is known by the agents, within which the agents share part of their histories according to some predefined rules (Nayyar et al., 2013b). The approach builds a new POMDP structure based solely on the information known to all agents, the common information (CI), with the new state being the state and all action/observation histories (AOHs), the new observation being the incoming CI, and the new action being the "prescription functions," which are full characterizations of how the agents should act based on their private information. Importantly, since the new structure only uses the CI, the computation can be done simultaneously by all agents, and the problem becomes a centralized POMDP, which can be solved using dynamic programming (DP). In effect, one can imagine that there is a single *virtual* coordinator who only knows the CI and uses this to perform all the computations, i.e., solve the resulting (centralized) DP, while the agents just follow the resulting prescriptions.

In contrast to its counterpart where the model is known, MARL faces numerous challenges both in practice and in theoretical analysis, including hardness to define the learning goals when the agents are not fully cooperative, the curse of dimensionality when the agents consider the whole action space which grows exponentially with the number of agents, the exploration-exploitation trade-off that already exists in single-agent RL and is further complicated in MARL, and the non-stationarity issue caused by simultaneous learning by the agents; usually convergence (to an equilibrium), stability (of the learning dynamics), and adaptation (to the other agents) are desired when designing learning strategies (Buşoniu et al., 2008; K. Zhang, 2019). In MARL problems, the "effective environment" of an agent not only depends on the underlying environment but also the policies of the other agents, and as the other agents update their policies, the effective environment is not stationary.

MARL algorithms can be split based on the level of how one agent models the other agents, which ranges from simply ignoring to having a *theory of mind*; the algorithms also vary in terms of their underlying information structure (Hernandez-Leal et al., 2019; K. Zhang, 2019). Some works in the literature assume a centralized controller is available, while some others consider a fully decentralized scenario where agents can communicate through a network. In the latter case, commonly known as networked MARL, in some literature sending parameter or gradient information is allowed (Kar et al., 2013; Zhang et al., 2018), which is similar to the setup of distributed optimization (Nedić and Ozdaglar, 2009), while agents only send a part of their own AOHs in other papers. On the other hand, in the ignoring approach, every agent just ignores the existence of other agents and learns independently, so that the algorithm may fail to converge (Claus and Boutilier, 1998). In the most recent literature, including the Parameterized Interactive POMDP model where a cognitive hierarchy of different levels is built (Wunder et al., 2011, 2012), agents reason though the theory of mind when observing others' actions (Foerster et al., 2019). These approaches resemble the person-by-person approach used in studying decentralized team problems in stochastic control. In certain cases, this approach can identify the structure of optimal policies which can then be combined with the designer approach to yield optimal performance, but typically these yield a Nash equilibrium that is not optimal; see Nayyar et al. (2013b, 2014); Witsenhausen (1973) for more details.

Naghizadeh et al. (2019) studies the impact of communication on the learning procedure through the Dec-POMDP example of a "matching problem" where two agents aim to match both of their actions to the current state; this is also the problem studied in this paper. However, instead of using policy gradient methods that do not work well in MARL problems (Foerster et al., 2019), we tackle the non-stationarity issue by envisaging that the learning is done by the virtual coordinator using all available CI by means of the CI approach pioneered by (Nayyar et al., 2013b, 2014) so that the environment remains stationary and DP can be applied (and a sequential decomposition holds as well). Moreover, instead of a cheap talk model, we consider costly communications so that the agents need to judiciously decide whether to communicate or not. In our setting, we critically use the facts that communication can increase the CI between agents, and furthermore, can also reduce complexity of the algorithms; without the increased CI agents will need to maintain long histories and the compression of this history via CI results in reduced complexity. Nayyar et al. (2013a) also studies the rule of communication in decentralized systems; in their model, an energy-limited sensor decides whether to send a costly communication containing the true state and its energy level to an estimator, which needs to estimate the state. Using the CI approach and assuming the model is known, they obtain a DP characterization. The existence of a communication decision creates CI – even the decision to not communicate conveys information to the estimator.

The idea of CI, sometimes termed as "common knowledge (CK)" in the RL literature, has been applied to MARL problems recently. MACKRL is proposed by de Witt et al. (2019), where the agents are partitioned into a hierarchical tree of subgroups within which they learn their policies, which are the probabilities of their joint actions conditioning on their common AOHs. However, in the paper the tree is predefined, limiting the information structure; furthermore, they do not exploit prescription functions, so that within a subgroup the agents' private AOHs are still wasted. In (Foerster et al., 2019), the public belief (basically the information state in (Nayyar et al., 2013b)), which is the probability of the private features given the public features, is proposed to construct a Public Belief MDP. They consider policies as a distribution, which is parameterized as a deep neural network, of prescription functions given the public belief and public features. They apply the method to the game of Habani, which is a cooperative partial-information card game that involves communications among players to reveal information. Later on in (Lerer et al., 2019) a technique called "Search for Partially Observing Teams of Agents" is further introduced to provide the state-of-the-art performance of Hanabi in two-player games. While these two papers show how to use the CI approach in practice, they do not provide regret characterization of their learning methods though nor is it clear if convergence to near-optimal policy is guaranteed by their methods.

Main Contributions: We demonstrate the effectiveness of the CI approach through a general version of the matching problem. We allow communication between the agents as a (costly) action and use this to obtain a two sub-step Dec-POMDP; that communication increases the CI between agents is critical to our analysis. When the model is known, we derive an optimal policy via a series of results for the underlying information structure using the CI approach. When the model is unknown, the structural results from the known-model setting give good guidance in designing a low-regret learning scheme, whose applicability is unclear without the simplifications provided by the common information approach. When the model (parameters) is unknown, we provide a reinforcement learning algorithm using a novel "two-stage hierarchical bandit" problem whose information structure fits the matching problem. Specifically, we propose a "two-stage Upper Confidence Bound (UCB) algorithm" that achieves asymptotic optimality by demonstrating that the algorithm achieves near-optimal regret. Furthermore, for a generalized model we show how Q-learning based schemes can yield optimal performance. In our work we also highlight the fact that a naive learning scheme that simply estimates the unknown kernel and then applies the optimal policy using the known-model analysis (i.e., certainty-equivalence based control), could fail as the decentralized setting could inhibit kernel estimation.

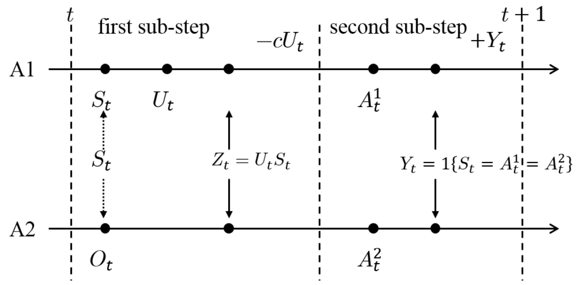

Figure 1: Time-line of states, observations, actions, and rewards.

2. Problem Formulation

Here we give the general decentralized POMDP (Dec-POMDP) formulation and the "matching problem" as an instance of two-step Dec-POMDP, and briefly summarize our solution approach.

The Dec-POMDP Model: For ease of presentation, we describe the case of two agents, A1 and A2, which is considered throughout this paper. A Dec-POMDP model is the following tuple $(\mathcal{S}, \mathcal{A}^1, \mathcal{A}^2, \mathbb{P}_T, R, \mathcal{O}^1, \mathcal{O}^2, \mathbb{P}_O, \lambda)$, where $\mathcal{S}$ is a state space, $\mathcal{A}^1$ and $\mathcal{A}^2$ are the action spaces of A1 and A2, $R(S, A^1, A^2)$ is the reward the agents get when they play $(A^1, A^2) \in \mathcal{A}^1 \times \mathcal{A}^2$ at current state $S \in \mathcal{S}$, $\mathbb{P}_T(S, A^1, A^2, S') = \mathbb{P}(S'|S, A^1, A^2)$ is the transition probability of the state transiting to $\mathcal{S}' \in \mathcal{S}$ when the agents play (A^1, A^2) at state S, $\mathcal{O}^1$ and $\mathcal{O}^2$ are the observation spaces of A1 and A2, $\mathbb{P}_O(S, A^1, A^2, S', O^1, O^2) = \mathbb{P}(O^1, O^2|S, A^1, A^2, S')$ is the probability of observing $(O^1, O^2) \in \mathcal{O}^1 \times \mathcal{O}^2$ of such transition, and λ is the discount factor (Bernstein et al., 2002). In some formulations (Nayyar et al., 2013b), the observations are just the output of the state going through noisy channels $\mathbb{P}_O(O^1, O^2|S)$ and are obtained before the agents take actions. However, the two formulations are equivalent since one could match them with dummy states and dummy transitions. They only differ slightly in the way of updating beliefs. Note that in RL literature the underlying Dec-POMDP model is usually assumed to be *time-invariant*, while in decentralized stochastic control theory the model could be time-varying. In our set-up we will assume a time-invariant model.

The Matching Problem: We assume A1 is the informed agent, and A2 is the uninformed agent. In the matching problem, which we note once again as being time-invariant, the state space, the observation spaces, and the action spaces are all the same $\mathcal{S} = \mathcal{A}^1 = \mathcal{A}^2 = \mathcal{O}^1 = \mathcal{O}^2 = \{+1, -1\}$, and both agents try to match their actions to the state to get rewards. At time 0 nature draws the initial value of the state variable S_1 with prior $\mathbb{P}(S_1 = +1) = 1 - \epsilon$ and $\mathbb{P}(S_1 = -1) = \epsilon$, which is known to both agents. As illustrated in Figure 1, each time step is split into two sub-steps. In the first sub-step, the following events occur sequentially:

- The state S_t evolves according to the kernel $\mathbb{P}(S_t = S_{t-1}|Y_{t-1} = 1) = \mathbb{P}(S_t = -S_{t-1}|Y_{t-1} = 0) = 1 - \epsilon$ and $\mathbb{P}(S_t = S_{t-1}|Y_{t-1} = 0) = \mathbb{P}(S_t = -S_{t-1}|Y_{t-1} = 1) = \epsilon$.
- A1 perfectly observes S_t, i.e., $O_t^1 = S_t$, while A2 partially observes O_t^2, where $\mathbb{P}(O_t^2 = S_t) = \beta$ and $\mathbb{P}(O_t^2 = -S_t) = 1 - \beta$. We will treat O_t^1 directly as S_t so that we can drop the superscript in O_t^2 and write O_t instead.
- A1 decides $U_t \in \{0, 1\}$, where 1 means to communicate and 0 means not.
- Both agents observe the communication result $Z_t = U_tS_t$. A1 knows what it communicated and so our assumption is that A2 gets the message without any errors if it was sent.

- They receive reward $-cU_t$ where $c \in [0, 1)$ reflects the communication cost.

In the second sub-step, the following events occur sequentially:

- The state makes a dummy change, i.e. remains in S_t.
- A1 decides its action A_t^1 and A2 decides its action A_t^2. This is done simultaneously and neither agent observes the action of the other agent.
- Both agents observe the coordinating result $Y_t \in \{0, 1\}$, where $Y_t(S_t, A_t) = \mathbf{1}\{S_t = A_t^1 = A_t^2\}$ and $A_t = (A_t^1, A_t^2)$.
- They receive reward Y_t.

The total reward for this step is then $R_t = R_t(S_t, A_t, U_t) = \mathbf{1}\{S_t = A_t^1 = A_t^2\} - cU_t = Y_t - cU_t$. The objective is to find the optimal control to maximize the expected long term discounted reward $\max_g \mathbb{E}_g \left[\sum_{t=1}^{\infty} \lambda^t R_t\right]$. All the parameters, including ϵ, β, c, and λ, are assumed to be common knowledge.

We assume both agents have perfect recall. Hence, just before A1 takes action in the first sub-step at time t, the agents' information structure is as follows: the CI is $(Z_{1:t-1}, Y_{1:t-1})$, A1's private information includes $(S_{1:t}, A_{1:t-1}^1)$, and A2's private information includes $(O_{1:t}, A_{1:t-1}^2)$. Note that both agents' private information can be strictly less that the variables indicated: communication by A1 reveals the system state to A2 and a reward of 1 reveals the state to A2 and the actions to both agents. Just before both agents take actions in the second sub-step at time t, the agents' information structure is as follows: the CI is $(Z_{1:t}, Y_{1:t-1})$, A1's private information includes $(S_{1:t}, A_{1:t-1}^1)$, and A2's private information includes $(O_{1:t}, A_{1:t-1}^2)$. For a random process $\{X_i\}_{i \in \mathbb{N}}$, we use the convention of $X_{1:t} = (X_1, \dots, X_t)$.

The matching problem could be seen as a "two-step Dec-POMDP model," where the environment flips between the two tuples $(\mathcal{S}^{\text{i}}, \mathcal{A}^{\text{i},1}, \mathcal{A}^{\text{i},2}, R^{\text{i}}, \mathcal{O}^{\text{i},1}, \mathcal{O}^{\text{i},2})$ and $(\mathcal{S}^{\text{ii}}, \mathcal{A}^{\text{ii},1}, \mathcal{A}^{\text{ii},2}, R^{\text{ii}}, \mathcal{O}^{\text{ii},1}, \mathcal{O}^{\text{ii},2})$. In the first sub-step, $\mathcal{S}^{\text{i}}$ corresponds to the state space $\mathcal{S}$ as above, $U_t \in \mathcal{A}^{\text{i},1}$ is the communication decision, and there is no action for A2 so that $\mathcal{A}^{\text{i},2} = \phi$, $R^{\text{i}} = -cU_t$, and $\mathcal{O}^{\text{i},1} = \mathcal{O}^{\text{i},2} = \{+1, 0, -1\}$ is the communication result. In the second sub-step, the state $(S_t, U_t) \in \mathcal{S}^{\text{ii}}$ now includes the previous communication decision, $\mathcal{A}^{\text{ii},1}$ and $\mathcal{A}^{\text{ii},2}$ correspond to the action spaces $\mathcal{A}^1$ and $\mathcal{A}^2$ as above, $R^{\text{ii}} = Y_t$, and $\mathcal{O}^{\text{ii},1}$ and $\mathcal{O}^{\text{ii},2}$ correspond to the observation spaces $\mathcal{O}^1$ and $\mathcal{O}^2$ *for the next time-step* as above.

Our model differs from that of Naghizadeh et al. (2019) in the following aspects. First, in their model the state transits deterministically, in which case agents may just apply the trivial policy of playing $+1$ at all times for both agents to achieve optimality; this corresponds to the case of $\epsilon = 0$ in our model. Second, in our model communication is costly but in Naghizadeh et al. (2019) there is not any. Since A1 has perfect observations, if at a given time-step it sends its observation to A2, then A2 can directly follow, and both will achieve the reward of 1 at that time-step. Therefore, if the communication costs nothing, then A1 could just send its observation in every time-step and the agents will get the maximum possible long-term reward; note that this is one simple means to achieve optimal performance with communications in Naghizadeh et al. (2019).

Solution Approach: In Section 3, we derive an optimal policy when the model is known using the CI approach proposed by Nayyar et al. (2013b, 2014). We obtain a series of structural results using the certain features of the information structure of the matching problem, and finally conclude that there exists a time-invariant optimal policy for the problem. We give a full characterization of the optimal policy in Theorem 5 and Theorem 8. We then use this knowledge to develop a MARL methodology in Section 4 when the parameters of the model is unknown. We also discuss three generalizations of the model in Section 5; we give the optimal policy for one of them, with its MARL counterpart when the model is unknown, which involves Q-learning.

3. Stochastic Control Viewpoint: Structural Results and an Optimal Policy

We start by introducing the notation. Then in Subsection 3.1, we apply the CI approach proposed by (Nayyar et al., 2013b, 2014) to the matching problem, determine the information state and the DP equations of the problem. The structural results rely on two important properties in our model: (i) whenever the two agents obtain the reward of 1, they know they correctly match the state in that time-step, and (ii) whenever A1 communicates the state to A2, then too both agents know the state before taking their action; in other words, there are times when the current state becomes CI. In Subsection 3.2, we exploit these two features to show that both the information states and the DP equations can be simplified. Finally, using the results and other properties of the matching problem, we characterize the optimal policy of the problem based on the parameters in Subsection 3.3.

Notation: We treat the two sub-steps as two separate time steps when writing the DP equation. We denote everything regarding the first and second sub-steps with superscripts $^{\text{i}}$ and $^{\text{ii}}$, respectively, but both with the same subscript t for the time-step index. When a sub-step index and agent index are used together, e.g. the second sub-step for A1, we write $^{\text{ii},1}$, that is, the sub-step index comes first. Denote the tuple of private information at time t as $M_t = (S_t, O_t, A_{t-1}) = (M_t^1, M_t^2)$, where $M_t^1 = (S_t, A_{t-1}^1)$ and $M_t^2 = (O_t, A_{t-1}^2)$ are the private information of A1 and A2, respectively; the M_1's do not contain the A components. Other notation includes the following: g represents a strategy, γ is a prescription function, Π is an information state, and V is the value function of an information state. For any variable X, we denote $\Omega(X)$ to be the appropriate space that X takes values in. All tuples are treated as unordered tuples in this paper. Usually an upper case variable is a random variable and the lower case version for that variable is a realization of the random variable.

3.1. The Common Information Approach

In the CI approach, the agents (or effectively the virtual coordinator) maintain the conditional probability of the private information and the state given the CI as the information state of the centralized POMDP, and solve the DP equation to find an optimal prescription function, which is an optimal strategy given the current information state. Based on this the first information state (in the first sub-step) is given by

$$\Pi_t^{\text{i}}(m^t) = \mathbb{P}^{g_{1:t-1}}(M_{1:t} = m^t | Z_{1:t-1}, Y_{1:t-1}), \tag{1}$$

a distribution over $\Omega(M_{1:t})$ given $Z_{1:t-1}$, $Y_{1:t-1}$, and the strategies $g_{1:t-1}$. Similarly, the second information state (in the second sub-step) is

$$\Pi_t^{\text{ii}}(m^t) = \mathbb{P}^{g_{1:t-1}, g_t^1}(M_{1:t} = m^t | Z_{1:t}, Y_{1:t-1}), \tag{2}$$

a distribution over $\Omega(M_{1:t})$ given $Z_{1:t}$, $Y_{1:t-1}$, $g_{1:t-1}$, and the strategy g_t^{i}. The strategy in the first sub-step maps A1's private information plus the information state to the communication decision

$$U_t = g_t^{\text{i}}(M_{1:t}^1, \Pi_t^{\text{i}}) := \gamma_t^{\text{i}}(M_{1:t}^1), \tag{3}$$

where the prescription $\gamma_t^{\rm i}$ is the strategy given the information state $\Pi_t^{\rm i}$. The strategies in the second sub-step map A1 and A2's private information plus the information state to their actions

$$\begin{aligned} A_t^1 &= g_t^{\rm ii,1}(M_{1:t}^1, \Pi_t^{\rm ii}) := \gamma_t^{\rm ii,1}(M_{1:t}^1), \\ A_t^2 &= g_t^{\rm ii,2}(M_{1:t}^2, \Pi_t^{\rm ii}) := \gamma_t^{\rm ii,2}(M_{1:t}^2), \end{aligned} \tag{4}$$

where the prescriptions for A1 and A2, $\gamma_t^{\rm ii,1}$ and $\gamma_t^{\rm ii,2}$, are again their strategies given the information state $\Pi_t^{\rm ii}$. The prescriptions belong to the function spaces $\gamma_t^{\rm i} \in \Omega(\Omega(S_{1:t}) \times \Omega(A_{1:t-1}^1) \to \Omega(U_t))$, $\gamma_t^{\rm ii,1} \in \Omega(\Omega(S_{1:t}) \times \Omega(A_{1:t-1}^1) \to \Omega(A_t^1))$, and $\gamma_t^{\rm ii,2} \in \Omega(\Omega(O_{1:t}) \times \Omega(A_{1:t-1}^2) \to \Omega(A_t^2))$. We write $g_t := (g_t^{\rm i}, g_t^{\rm ii}) := (g_t^{\rm i}, g_t^{\rm ii,1}, g_t^{\rm ii,2})$ where $g_t^{\rm ii} := (g_t^{\rm ii,1}, g_t^{\rm ii,2})$ and also the corresponding γ's.

The DP equations can then be written as

$$\begin{aligned} V_t^{\rm i}(\pi_t^{\rm i}) &= \sup_{\gamma_t^{\rm i}} \mathbb{E}\left[-c\gamma_t^{\rm i}(M_{1:t}) + V_t^{\rm ii}(\eta_t^{\rm i}(\Pi_t^{\rm i}, \gamma_t^{\rm i}, Z_t)) \middle| \Pi_t^{\rm i} = \pi_t^{\rm i} \right], \\ V_t^{\rm ii}(\pi_t^{\rm ii}) &= \sup_{\gamma_t^{\rm ii}} \mathbb{E}\left[Y_t(S_t, \gamma_t^{\rm ii}(M_{1:t})) + \lambda V_{t+1}^{\rm i}(\eta_t^{\rm ii}(\Pi_t^{\rm ii}, \gamma_t^{\rm ii}, Y_t)) \middle| \Pi_t^{\rm ii} = \pi_t^{\rm ii} \right], \end{aligned} \tag{5}$$

where $V_t^{\rm i}(\pi_t^{\rm i})$ and $V_t^{\rm ii}(\pi_t^{\rm ii})$ are the value functions under an optimal policy for the information states in the first and the second sub-steps, and $\eta_t^{\rm i}(\Pi_t^{\rm i}, \gamma_t^{\rm i}, Z_t)$ and $\eta_t^{\rm ii}(\Pi_t^{\rm ii}, \gamma_t^{\rm ii}, Y_t)$ are the update rules for information states using Bayes rule namely,

$$\begin{aligned} &\Pi_t^{\rm ii}(m^t) = \eta_t^{\rm i}(\Pi_t^{\rm i}, \gamma_t^{\rm i}, Z_t)(m^t) = \mathbb{P}(M_{1:t} = m^t | \Pi_t^{\rm i}, \gamma_t^{\rm i}, Z_{1:t}, Y_{1:t-1}) \\ &= \frac{\mathbb{P}(M_{1:t} = m^t | \Pi_t^{\rm i}, \gamma_t^{\rm i}, Z_{1:t-1}, Y_{1:t-1}) \mathbb{P}(Z_t | M_{1:t} = m^t, \Pi_t^{\rm i}, \gamma_t^{\rm i}, Z_{1:t-1}, Y_{1:t-1})}{\sum_{m^{t\prime}} \mathbb{P}(M_{1:t} = m^{t\prime} | \Pi_t^{\rm i}, \gamma_t^{\rm i}, Z_{1:t-1}, Y_{1:t-1}) \mathbb{P}(Z_t | M_{1:t} = m^{t\prime}, \Pi_t^{\rm i}, \gamma_t^{\rm i}, Z_{1:t-1}, Y_{1:t-1})} \\ &= \frac{\Pi_t^{\rm i}(m^t) \mathbb{P}(Z_t | M_{1:t} = m^t, \gamma_t^{\rm i})}{\sum_{m^{t\prime}} \Pi_t^{\rm i}(m^{t\prime}) \mathbb{P}(Z_t | M_{1:t} = m^{t\prime}, \gamma_t^{\rm i})}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\Pi_{t+1}^{\rm i}(m^{t+1}) = \eta_t^{\rm ii}(\Pi_t^{\rm ii}, \gamma_t^{\rm ii}, Y_t)(m^{t+1}) = \mathbb{P}(M_{1:t+1} = m^{t+1} | \Pi_t^{\rm ii}, \gamma_t^{\rm ii}, Z_{1:t}, Y_{1:t}) \\ &= \frac{\mathbb{P}(M_{1:t} = m^t | \Pi_t^{\rm ii}, \gamma_t^{\rm ii}, Z_{1:t}, Y_{1:t-1}) \mathbb{P}(M_{t+1} = m_{t+1}, Y_t | M_{1:t} = m^t, \Pi_t^{\rm ii}, \gamma_t^{\rm ii}, Z_{1:t}, Y_{1:t-1})}{\sum_{m^{t+1\prime}} \mathbb{P}(M_{1:t} = m^{t\prime} | \Pi_t^{\rm ii}, \gamma_t^{\rm ii}, Z_{1:t}, Y_{1:t-1}) \mathbb{P}(M_{t+1} = m_{t+1}{}', Y_t | M_{1:t} = m^{t\prime}, \Pi_t^{\rm ii}, \gamma_t^{\rm ii}, Z_{1:t}, Y_{1:t-1})} \\ &= \frac{\Pi_t^{\rm ii}(m^t) \mathbb{P}(M_{t+1} = m_{t+1}, Y_t | M_{1:t} = m^t, \gamma_t^{\rm ii})}{\sum_{m^{t+1\prime}} \Pi_t^{\rm ii}(m^{t\prime}) \mathbb{P}(M_{t+1} = m_{t+1}{}', Y_t | M_{1:t} = m^{t\prime}, \gamma_t^{\rm ii})}. \end{aligned} \tag{7}$$

3.2. Structural Results When the Information State Collapses

Notice that once some non-zero Z or Y appears, knowing the current value of S in the CI collapses the beliefs at the time-step (i.e. becomes a deterministic value); it is critical to use only the CI to coordinate the actions of the agents. The following lemma then says that to find an optimal strategy, it is sufficient to ignore all the history of private information prior to the time-step.

Lemma 1 *At time step t, define*

$$\begin{aligned} \bar{t}^{\rm i}(Z_{1:t-1}, Y_{1:t-1}) &= \max\{t' : Z_{t'} \neq 0 \textit{ for } 1 \leq t' < t,\ \textit{or } Y_{t'} \neq 0 \textit{ for } 1 \leq t' < t\}, \\ \bar{t}^{\rm ii}(Z_{1:t}, Y_{1:t-1}) &= \max\{t' : Z_{t'} \neq 0 \textit{ for } 1 \leq t' \leq t,\ \textit{or } Y_{t'} \neq 0 \textit{ for } 1 \leq t' < t\}. \end{aligned} \tag{8}$$

Suppose the sets over which the maxima are take are not empty, so that $\bar{t}^{\mathrm{i}}$ and $\bar{t}^{\mathrm{ii}}$ are well-defined. Then there exist $\gamma_t^{\mathrm{i}*} \in \Omega(\Omega(S_{\bar{t}^{\mathrm{i}}+1:t}) \times \Omega(A^1_{\bar{t}^{\mathrm{i}}:t-1}) \to \Omega(U_t))$ and $\gamma_t^{\mathrm{ii}*} \in \Omega(\Omega(M_{\bar{t}^{\mathrm{ii}}+1:t}) \to \Omega(A_t))$ that achieve the optimality of $V_{t-1}^{\mathrm{i}}(\pi_t^{\mathrm{i}})$ and $V_t^{\mathrm{ii}}(\pi_t^{\mathrm{ii}})$, respectively. In other words, it is without loss of generality to restrict our attention to the above two function spaces instead of $\Omega(\Omega(S_{1:t}) \times \Omega(A^1_{1:t-1}) \to \Omega(U_t))$ and $\Omega(\Omega(M_{1:t}) \to \Omega(A_t))$ for the decisions of γ_t^{i} and γ_t^{ii}, respectively.

Proof We only prove the case for the first sub-step. The proof of the second sub-step is the same. Notice that whenever $Z_t \neq 0$ or $Y_t \neq 0$, S_t is in the CI – if $Z_t = U_t S_t \neq 0$, it means $U_t = 1$ and $Z_t = S_t$, and since Z_t is CI, so is S_t; on the other hand, if $Y_t = \mathbf{1}\{S_t = A_t^1 = A_t^2\} = 1 \neq 0$, then both agents can deduce S_t from their private information, namely past actions A_t^1 and A_t^2.

Since we consider a Markovian model, given a strategy over time, $S_{\bar{t}^{\mathrm{i}}}$ will be the sufficient statistic of all variables prior to $\bar{t}^{\mathrm{i}}$. Consider an optimal prescription $\gamma_t^{\mathrm{i}*}$ in the original space and two admissible histories $m_1, m_2 \in \Omega(S_{1:\bar{t}^{\mathrm{i}}-1}) \times \Omega(A^1_{1:\bar{t}^{\mathrm{i}}-1})$ and $m_3 \in \Omega(S_{\bar{t}^{\mathrm{i}}:t}) \times \Omega(A^1_{\bar{t}^{\mathrm{i}}:t-1})$ so that both (m_1, m_3) and (m_2, m_3) are consistent with the CI. We use proof by contradiction that under $\gamma_t^{\mathrm{i}*}$, the two histories (m_1, m_3) and (m_2, m_3) must have the same value. One can then construct an optimal strategy that disregards the first part of the tuple.

For more details see Appendix A.1. ■

Remark 2 *At time step t, $\bar{t}^{\mathrm{i}}$ is at most $t-1$; since the agents have observed $Y_{\bar{t}^{\mathrm{i}}}$, they can deduce each others' actions as $S_{\bar{t}^{\mathrm{i}}}$ has become CI, so that $A_{\bar{t}^{\mathrm{i}}}$ also becomes CI. This is also the case for the second sub-step when $\bar{t}^{\mathrm{ii}} \leq t-1$. When $\bar{t}^{\mathrm{ii}} = t$, A1 just sent S_t to A2, and A_t is yet to be decided.*

After Lemma 1 is established, whenever some non-zero Z or Y appears, we will restrict the spaces of the prescriptions according to Lemma 1. Then the information states can be further split. Define two new information states for the two sub-steps as

$$\begin{aligned}\Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{i}}+1:t}(m) &= \mathbb{P}^{g_{\bar{t}^{\mathrm{i}}:t-1}}(M_{\bar{t}^{\mathrm{i}}+1:t} = m | S_{\bar{t}^{\mathrm{i}}}, Z_{\bar{t}^{\mathrm{i}}:t-1}, Y_{\bar{t}^{\mathrm{i}}:t-1}),\\ \Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t}(m) &= \mathbb{P}^{g_{\bar{t}^{\mathrm{ii}}:t-1}, g_t^{\mathrm{i}}}(M_{\bar{t}^{\mathrm{ii}}+1:t} = m | S_{\bar{t}^{\mathrm{ii}}}, Z_{\bar{t}^{\mathrm{ii}}:t}, Y_{\bar{t}^{\mathrm{ii}}:t-1}).\end{aligned} \tag{9}$$

Lemma 3 *Consider time step t. Suppose we use the prescription space reduction of Lemma 1 throughout and that $\bar{t}^{\mathrm{i}}$ is well-defined. Then given the CI $(Z_{1:t-1}, Y_{1:t-1})$, Π_t^{i} can be split as*

$$\Pi_t^{\mathrm{i}}(m, m') = \Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{i}}}(m) \cdot \Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{i}}+1:t}(m'), \tag{10}$$

where $m \in \Omega(M_{1:\bar{t}^{\mathrm{i}}})$ and $m' \in \Omega(M_{\bar{t}^{\mathrm{i}}+1:t})$. Similarly, given the CI $(Z_{1:t}, Y_{1:t-1})$ and the assumption that $\bar{t}^{\mathrm{ii}}$ is well-defined, Π_t^{ii} can be split as

$$\Pi_t^{\mathrm{ii}}(m, m') = \Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}}(m) \cdot \Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t}(m'), \tag{11}$$

where $m \in \Omega(M_{1:\bar{t}^{\mathrm{ii}}})$ and $m' \in \Omega(M_{\bar{t}^{\mathrm{ii}}+1:t})$.

Proof The proof follows from the definition of conditional probability, the fact that $S_{\bar{t}^{\mathrm{i}}}$ is the sufficient statistics for the previous variables given the strategies, and Lemma 1 so that the strategies do not depend on the first part of the history tuple. See Appendix A.2. ■

An intermediate proposition given in Appendix A.3, Proposition 11, use the strategy space reduction in Lemma 1 and the split of information state in Lemma 3 to rewrite the DP equations and the information state updating rules in (6) and (7). The following proposition then follows from the stationarity of the model and the fact that the dynamics in Proposition 11 do not depend on the values of $\bar{t}^{\mathrm{i}}$ and $\bar{t}^{\mathrm{ii}}$ but only on the differences between t and them.

Proposition 4 *Consider time step t. Given the CI $(Z_{1:t-1}, Y_{1:t-1})$, define $\tau^{\mathrm{i}} = t - \bar{t}^{\mathrm{i}}$. The DP equation for the first sub-step can be written as*

$$V_t^{\mathrm{i}}(\pi_{\tau^{\mathrm{i}}}^{\mathrm{i}}) = \sup_{\gamma_{\tau^{\mathrm{i}}}^{\mathrm{i}}} \mathbb{E}\left[-c\gamma_{\tau^{\mathrm{i}}}^{\mathrm{i}}(M_{\bar{t}^{\mathrm{i}}+1:t}) + V_t^{\mathrm{ii}}(\eta_{\tau^{\mathrm{i}}}^{\mathrm{i}}(\Pi_{\tau^{\mathrm{i}}}^{\mathrm{i}}, \gamma_{\tau^{\mathrm{i}}}^{\mathrm{i}}, Z_t)) \middle| \Pi_{\tau^{\mathrm{i}}}^{\mathrm{i}} = \pi_{\tau^{\mathrm{i}}}^{\mathrm{i}}\right], \tag{12}$$

where $\pi_{\tau^{\mathrm{i}}}^{\mathrm{i}}$ and $\gamma_{\tau^{\mathrm{i}}}^{\mathrm{i}}$ now only depend on the gap between $\bar{t}^{\mathrm{i}}$ and t. Similarly, given the CI $(Z_{1:t}, Y_{1:t-1})$, define $\tau^{\mathrm{ii}} = t - \bar{t}^{\mathrm{ii}}$. The DP equation for the second sub-step can be written as

$$V_t^{\mathrm{ii}}(\pi_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}) = \sup_{\gamma_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}} \mathbb{E}\left[Y_t(S_t, \gamma_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}(M_{\bar{t}^{\mathrm{ii}}+1:t})) + \lambda V_{t+1}^{\mathrm{i}}(\eta_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}(\Pi_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}, \gamma_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}, Y_t)) \middle| \Pi_{\tau^{\mathrm{ii}}}^{\mathrm{ii}} = \pi_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}\right], \tag{13}$$

where $\pi_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}$ and $\gamma_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}$ now only depend on the gap between $\bar{t}^{\mathrm{ii}}$ and t. The terms $\eta_{\tau^{\mathrm{i}}}^{\mathrm{i}}$ and $\eta_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}$ are the update rules[1].

Proof This follows from the stationarity of the model and the fact that the dynamics in Proposition 11 do not depend on the values of $\bar{t}^{\mathrm{i}}$ and $\bar{t}^{\mathrm{ii}}$ but only on the differences between t and them. See Appendix A.4. ■

3.3. Characterization of the Optimal Policy

Note that Proposition 4 follows directly by the fact that $S_{\bar{t}}$ is deterministic in CI and the time-invariance of the model. In this Subsection, we use the details of the model to derive the optimal policy. Note that since the DP equations are time-invariant, equations (12) and (13) do not directly depend on t; in particular, V_t is actually just V, and (Z_t, Y_t) as well as $M_{\bar{t}+1:t}$ can be viewed as "the current (Z, Y)" and "the most recent τ M's." Moreover, now that we have Proposition 4, we can review (9) again. First, from the definition of $\bar{t}^{\mathrm{i}}$ and $\bar{t}^{\mathrm{ii}}$, the CIs $Z_{\bar{t}^{\mathrm{i}}+1:t-1}, Y_{\bar{t}^{\mathrm{i}}+1:t-1}$ and $Z_{\bar{t}^{\mathrm{ii}}+1:t}, Y_{\bar{t}^{\mathrm{ii}}+1:t-1}$ are all 0's, so that it is unnecessary to put them in the condition. Second, from the time-invariance of Proposition 4, the value of t does not matter. We can adopt another index system, where 0 is the current time step, and $-t$ is the t-th most recent time step. Using this indexing and Proposition 4, (9) can be rewritten as

$$\begin{aligned} \Pi_{\tau^{\mathrm{i}}}^{\mathrm{i}}(m) &= \mathbb{P}^{g_{-\tau^{\mathrm{i}}:-1}}(M_{-\tau^{\mathrm{i}}+1:0} = m | S_{-\tau^{\mathrm{i}}}, Z_{-\tau^{\mathrm{i}}}, Y_{-\tau^{\mathrm{i}}}), \\ \Pi_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}(m) &= \mathbb{P}^{g_{-\tau^{\mathrm{ii}}:-1}, g_0^{\mathrm{i}}}(M_{-\tau^{\mathrm{ii}}+1:0} = m | S_{-\tau^{\mathrm{ii}}}, Z_{-\tau^{\mathrm{ii}}}, Y_{-\tau^{\mathrm{ii}}}). \end{aligned} \tag{14}$$

If $\tau^{\mathrm{ii}} = 0$, then Y_0 is yet to be observed and should not be present in $\Pi_0^{\mathrm{ii}}(m)$. The distribution $\Pi_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}$ is a *random* distribution whose realization only depends on the random variables $(S_{-\tau^{\mathrm{ii}}}, Z_{-\tau^{\mathrm{ii}}}, Y_{-\tau^{\mathrm{ii}}})$. We denote $\pi_{\tau^{\mathrm{ii}},s,v}^{\mathrm{ii}}$ as a realization of $\Pi_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}$, where $s \in \{+1, -1\}$ is the realization of $S_{-\tau^{\mathrm{ii}}}$, and $v = y$ if $Y_{-\tau^{\mathrm{ii}}} = 1$ and $v = z$ otherwise (in this case it must be that $Z_{-\tau^{\mathrm{ii}}} \neq 0$). As stated previously, when $\tau^{\mathrm{ii}} = 0$, it must be that $v = z$.

1. The update rules $\eta_{\tau^{\mathrm{i}}}^{\mathrm{i}}$ and $\eta_{\tau^{\mathrm{ii}}}^{\mathrm{ii}}$ are omitted for brevity. They follow from (24) and (26), except that now they do not depend on the values of $\bar{t}^{\mathrm{i}}$ and $\bar{t}^{\mathrm{ii}}$, but only on the differences between $\bar{t}^{\mathrm{i}}$ and t and between $\bar{t}^{\mathrm{ii}}$ and t, and hence, we index the differences by τ^{i} and τ^{ii}, respectively.

Theorem 5 *All optimal strategies, $g^*_{1:\infty}$, always assign $A^1_t = g^{\text{ii},1}_t(M^1_{1:t}, \Pi^{\text{ii}}_t) = S_t$ for all time steps t given the reward structure of $R^{\text{ii}}_t = Y_t(S_t, A_t) = \mathbf{1}\{S_t = A^1_t = A^2_t\}$.*

Proof A complete policy is of the form $g_{1:\infty} = (g^{\text{i}}_{1:\infty}, g^{\text{ii},1}_{1:\infty}, g^{\text{ii},2}_{1:\infty})$. Denote the policy that A1 always matches the state for all t as $g^{\text{ii},1\sharp}_{1:\infty}$. Under this policy, we have Corollary 6, that is, the previous state is always CI; moreover, for the remaining parts of the complete policy, we can find an optimal one, denoted as $(g^{\text{i}\sharp}_{1:\infty}, g^{\text{ii},2\sharp}_{1:\infty})$. We show that A1 will not deviated from $g^{\sharp}_{1:\infty} = (g^{\text{i}\sharp}_{1:\infty}, g^{\text{ii},1\sharp}_{1:\infty}, g^{\text{ii},2\sharp}_{1:\infty})$ by not matching the state. Hence, by the Policy Improvement Theorem (Sutton and Barto, 2018), $g^{\sharp}_{1:\infty}$ is an optimal policy. See Appendix B.1. ■

Since in our model the state is binary, Theorem 5 has important implications discussed below.

Corollary 6 *In the case where there are only two elements in the state space, under an optimal strategy over time, for any time step t, S_{t-1} is CI, so that in the first sub-step we always have the information state being Π^{i}_1, and in the second sub-step we always have the information state being either Π^{ii}_1 when A1 did not communicate or Π^{ii}_0 when it did.*

Proof A2 can deduce S_{t-1} by $S_{t-1} = A^2_{t-1} \cdot (-1)^{Y_{t-1}+1}$. Proposition 4 follows with $\vec{t}^{\text{i}} = t-1$, and $\vec{t}^{\text{ii}} = t-1$ if $U_t = 0$ and $\vec{t}^{\text{ii}} = t$ if $U_t = 1$. See Appendix B.2. ■

Remark 7 *Using Lemma 1, Corollary 6 implies the optimal strategy (or strategies) is* stationary.

Theorem 8 *There is an optimal strategy over time such that for all time step t, A1 follows the communication protocol (breaking ties arbitrarily if equality holds in all the inequalities)*

$$U_t = g^{\text{i}}(S_{t-1}, Y_{t-1}, S_t) = \begin{cases} 1, & \text{if } U'_t = 1, S_t = S_{t-1}\cdot(-1)^{Y_{t-1}+1}, \text{ and } \epsilon < 0.5, \\ 1, & \text{if } U'_t = 1, S_t = S_{t-1}\cdot(-1)^{Y_{t-1}}, \text{ and } \epsilon > 0.5, \\ 0, & \text{if } U'_t = 1, S_t = S_{t-1}\cdot(-1)^{Y_{t-1}}, \text{ and } \epsilon < 0.5, \\ 0, & \text{if } U'_t = 1, S_t = S_{t-1}\cdot(-1)^{Y_{t-1}+1}, \text{ and } \epsilon > 0.5, \\ 0, & \text{if } U'_t = 0, \end{cases} \tag{15}$$

where $\bar{\epsilon} = 1-\epsilon$ and $\bar{\beta} = 1-\beta$, and U'_t is determined by the model parameters (breaking ties arbitrarily)

$$U'_t = \begin{cases} 1, & \text{if } \epsilon < \beta,\ \ \epsilon+\beta > 1, \text{ and } \bar{\beta} > \min\{\epsilon,\bar{\epsilon}\}c, \\ 0, & \text{if } \epsilon < \beta,\ \ \epsilon+\beta > 1, \text{ and } \bar{\beta} < \min\{\epsilon,\bar{\epsilon}\}c, \\ 1, & \text{if } \epsilon < \beta,\ \ \epsilon+\beta < 1, \text{ and } \epsilon > \min\{\epsilon,\bar{\epsilon}\}c, \\ 0, & \text{if } \epsilon < \beta,\ \ \epsilon+\beta < 1, \text{ and } \epsilon < \min\{\epsilon,\bar{\epsilon}\}c, \\ 1, & \text{if } \epsilon > \beta,\ \ \epsilon+\beta > 1, \text{ and } \bar{\epsilon} > \min\{\epsilon,\bar{\epsilon}\}c, \\ 0, & \text{if } \epsilon > \beta,\ \ \epsilon+\beta > 1, \text{ and } \bar{\epsilon} < \min\{\epsilon,\bar{\epsilon}\}c, \\ 1, & \text{if } \epsilon > \beta,\ \ \epsilon+\beta < 1, \text{ and } \beta > \min\{\epsilon,\bar{\epsilon}\}c, \\ 0, & \text{if } \epsilon > \beta,\ \ \epsilon+\beta < 1, \text{ and } \beta < \min\{\epsilon,\bar{\epsilon}\}c. \end{cases} \tag{16}$$

In addition, if $U'_t = 1$, A2 decodes the received communication result U_t by (breaking ties arbitrarily but should follow A1's choice)

$$A^2_t = g^{\text{ii},2}(S_{t-1}, Y_{t-1}, O_t, U_t) = \begin{cases} S_{t-1}\cdot(-1)^{Y_{t-1}+1}, & \text{if } U_t = 1 \text{ and } \epsilon < 0.5, \\ S_{t-1}\cdot(-1)^{Y_{t-1}}, & \text{if } U_t = 1 \text{ and } \epsilon > 0.5, \\ S_{t-1}\cdot(-1)^{Y_{t-1}}, & \text{if } U_t = 0 \text{ and } \epsilon < 0.5, \\ S_{t-1}\cdot(-1)^{Y_{t-1}+1}, & \text{if } U_t = 0 \text{ and } \epsilon > 0.5; \end{cases} \tag{17}$$

otherwise, if $U'_t = 0$, then A2 chooses the best posterior guess according to (breaking ties arbitrarily)

$$A_t^2 = g^{\mathrm{ii},2}(S_{t-1}, Y_{t-1}, O_t, U_t) = \begin{cases} O_t, & \text{if } O_t = S_{t-1} \cdot (-1)^{Y_{t-1}+1} \text{ and } \epsilon < \beta, \\ -O_t, & \text{if } O_t = S_{t-1} \cdot (-1)^{Y_{t-1}+1} \text{ and } \epsilon > \beta, \\ O_t, & \text{if } O_t = S_{t-1} \cdot (-1)^{Y_{t-1}} \text{ and } \epsilon + \beta > 1, \\ -O_t, & \text{if } O_t = S_{t-1} \cdot (-1)^{Y_{t-1}} \text{ and } \epsilon + \beta < 1. \end{cases} \tag{18}$$

Proof Now that we have Corollary 6 and Remark 7 as well as the symmetries of transition and observation, all we have to do is to find a policy of a time step that maximizes the expected instantaneous reward. Performing the policy in every time step will maximize the expected long term reward.

Depending on the cost of communication c, A1 decides whether to communicate or not. If it does ($U'_t = 1$), it uses the 1-bit information of U_t to convey whether $S_t = \bar{S}_t = S_{t-1} \cdot (-1)^{Y_{t-1}+1}$ or $S_t = -\bar{S}_t = S_{t-1} \cdot (-1)^{Y_{t-1}}$. Naturally, it chooses the convention (a bijection from 0/1 to $\bar{S}_t$/$-\bar{S}_t$) so that 1 is sent less frequently, since sending 1 is costly. In particular, A1 is choosing between three policies: $g^{\mathrm{i}}[0,0]$, $g^{\mathrm{i}}[0,1]$, and $g^{\mathrm{i}}[1,0]$, where $g^{\mathrm{i}}[j,k]$ is defined to be the policy that A1 plays $U_t = j$ deterministically when $S_t = \bar{S}_t$ and $U_t = k$ deterministically when $S_t = -\bar{S}_t$, where $j, k \in \{0, 1\}$. The decision $U'_t = 1$ corresponds to the case where A1 chooses one of $g^{\mathrm{i}}[0,1]$ and $g^{\mathrm{i}}[1,0]$, and the decision $U'_t = 0$ corresponds to the case where A1 chooses $g^{\mathrm{i}}[0,0]$.

When A1 decides not to communicate at all ($U'_t = 0$), the expected instantaneous reward is $\mathbb{P}(Y_t = 1)$. There will be two cases $O_t = S_{t-1} \cdot (-1)^{Y_{t-1}+1}$ or $O_t = S_{t-1} \cdot (-1)^{Y_{t-1}}$; A2 maximizes $\mathbb{P}(Y_t = 1)$ by calculating the posterior probability of $\mathbb{P}(Y_t = 1 | A_t^2 = O_t)$ and $\mathbb{P}(Y_t = 1 | A_t^2 = -O_t)$ to decide whether to *follow or flip* its observation for each case. See Appendix B.3. ■

4. The Reinforcement Learning Solution

In this section we propose a reinforcement learning algorithm that achieves optimality asymptotically for the model of the matching problem described in Section 2 when the parameters are unknown (however, c is known since it can be directly observed). An initial thought would be for the agents to collect the histories and to *estimate* the parameters, while in each time step they just act based on Theorem 5 and Theorem 8 with the estimated parameters. However, the matching problem is one of the examples such that this "kernel-based method" will not work. In particular, O_t is the private information of A2, and although A1 can deduce A_t^2 in the next time step by Y_t, it has no means to tell O_t since it does not know whether A2 was following or flipping the observation. Without the knowledge of $O_{1:\infty}$, A1 cannot estimate β and enact the optimal communication decision in Theorem 8.

Notice that Theorem 5 does not depend on any parameter; hence, even if the parameters are unknown, A1 still always follows the state, so that S_{t-1} is still CI as in Corollary 6. Given S_{t-1}, the statistics of (S_t, O_t) the agents are facing is time-invariant. Since they are encountering a set of choices with fixed but unknown statistics, this belongs to the class of multi-agent stochastic multi-armed bandit (MAB) problems. In particular, beside A1's following the state, in every time-step, A1 first faces the decision of communicating or not and if so the decision of the communication protocol as in Theorem 8, then conditioning on A1 deciding *not to* communicate, A2 faces the decision of following or flipping its observation as in Theorem 8. A2 is actually facing two sets of decisions, depending on whether $O_t = \bar{S}_t$ or $O_t = -\bar{S}_t$, where $\bar{S}_t = S_{t-1} \cdot (-1)^{Y_{t-1}+1}$. In addition, since A1 does not have the information of O_t, it does not know which set of decisions A2 faced and A2's decision of follow/flip at the end of the time step. We formulate this as a hierarchical bandit problem with a specific information structure, which we call the "two-stage hierarchical bandit" (2HB) problem; see Protocol 1 below. To the best of our knowledge, this problem has not been studied in the literature before.

Protocol 1 Two-stage hierarchical bandit

input: K (#actions of A1), L (#actions of A2), $\{\mu_{k,l}\}_{k\in[K],l\in[L]}$ (unknown to the agents).
for $t = 1, \dots, T$ **do**
  A1 chooses $k_t \in [K]$.
  After receiving k_t, A2 chooses $l_t \in [L]$.
  A1 and A2 receive the reward $R_t(k_t, l_t)$, which is sampled from $Ber(\mu_{k_t,l_t})$.
end

In particular, A2 knows A1's choice of k_t, but A1 does not know A2's choice of l_t. If A1 knew l_t as well, this would be equivalent to the usual stochastic MAB problem with KL arms where the decision-making is centralized. In the matching problem, we can easily argue that we have two 2HB problems, one for $O_t = \bar{S}_t$ and one for $O_t = -\bar{S}_t$, selected by nature. Within each problem, A1 decides $g^{\text{i}} \in \{g^{\text{i}}[0,0], g^{\text{i}}[0,1], g^{\text{i}}[1,0]\}$ followed by A2 deciding $A_t^2 \in \{O_t, -O_t\}$. Therefore, $K = 3$ and $L = 2$ in our matching problem[2]. In the case of $O_t = \bar{S}_t$, there are four possible combinations of (S_{t-1}, Y_{t-1}, O_t); however, Theorem 8 implies that given the model parameters, the optimal policy will be either follow or flip the observations for all four combinations. Same in the case of $O_t = -\bar{S}_t$. These hold due to the symmetries of the state transition model and the observation model[3].

In Algorithm 1 below, we propose a modified UCB algorithm (Agrawal, 1995; Auer et al., 2002) for the 2HB problem, and show it achieves near-optimal regret bounds similar to those of UCB for both gap-independent and gap-dependent cases. At time t, define $\hat{\mu}_t^1(k) = \frac{1}{n_t^1(k)} \sum_{s=1}^{t-1} R_s(k_s, l_s) \mathbf{1}\{k_s = k\}$ to be the empirical mean of the k-th arm in the first level based on it being played $n_t^1(k) = \sum_{s=1}^{t-1} \mathbf{1}\{k_s = k\}$ times until $t-1$, and define $\hat{\mu}_t^2(k,l) = \frac{1}{n_t^2(k,s)} \sum_{s=1}^{t-1} R_s(k_s, l_s) \mathbf{1}\{k_s = k, l_s = l\}$ to be the empirical mean of the (k,l)-th arm based on it being played $n_t^2(k,l) = \sum_{s=1}^{t-1} \mathbf{1}\{k_s = k, l_s = l\}$ times up until $t-1$.

Despite both agents learning simultaneously in Algorithm 1, A1 explores arms more via a higher UCB so that A2 can discover the best arm conditioned on A1's arm being fixed. This allows us to achieve close-to-optimal regret as summarized in the theorems below. For ease of presentation we assume without loss of generality that the best arm of A2 given any A1's choice is the first, i.e., $\mu_{k,1} \geq \mu_{k,l} \ \forall \ \ k \in [K], l \in [L]$, and the best arm of A1 is the first too, i.e., $\mu_{1,1} \geq \mu_{k,1} \ \forall \ \ k \in [K]$.

2. When $g^{\text{i}} \in \{g^{\text{i}}[0,1], g^{\text{i}}[1,0]\}$, we can equivalently model the situation as the agents receiving $1 - \min\{\epsilon, \bar{\epsilon}\}c$ regardless of the value of A_t^2. Note that in our original description A2 has no other option but to align A_t^2 with the state indicated by A1's communication signal U_t.

3. It might be confusing why in the matching problem A2 receives A1's choice at first. In fact, it does not really receive the choice – if A1 chooses $g^{\text{i}}[0,1]$ but transmits a 0 following the communication protocol, then A2 only observes 0; even if A2 observes 1, it could not differentiate between $g^{\text{i}}[0,1]$ and $g^{\text{i}}[1,0]$. However, the fact that A2 receives k_t is not crucial, since A1's perspective is the same as that of the coordinator. Every computation needed in the following Algorithm 1 for A1 to obtain k_t can be done at A2 as well, so A2 knows the g^{i} A1 will be choosing in advance, but not the state S_t which only A1 can see.

Algorithm 1 two-stage UCB

input: K (#actions of A1), L (#actions of A2).

for $t = 1, \ldots, KL$ **do**

  A1 and A2 play each combination of arms (k, l) once, where $k \in [K]$ and $l \in [L]$.

end

for $t = KL + 1, \ldots, T$ **do**

  A1 chooses $k_t = \arg\max_{k \in [K]} \; \hat{\mu}_t^1(k) + 11\sqrt{\frac{L\log(LT/\delta)}{n_t^1(k)}}$.

  A2 chooses $l_t = \arg\max_{l \in [L]} \; \hat{\mu}_t^2(k_t, l) + 2\sqrt{\frac{\log(LT/\delta)}{n_t^2(k_t,l)}}$.

end

Theorem 9 *For the two-stage bandit problem, with probability of at least* $1 - \delta$*, Algorithm 1 achieves the regret bound of*

$$\sum_{t=1}^{T} \mu_{1,1} - R_t(k_t, l_t) \leq O\left(\sqrt{KLT\log(LT/\delta)}\right). \tag{19}$$

Proof See Appendix C.2. ■

Theorem 10 *For the two-stage bandit problem, with probability of at least* $1 - \delta$*, Algorithm 1 achieves the regret bound of*

$$\sum_{t=1}^{T} \mu_{1,1} - R_t(k_t, l_t) \leq L \sum_{k \in [K]\setminus\{1\}} O\left(\frac{\log(LT/\delta)}{\mu_{1,1} - \mu_{k,1}}\right) + \sum_{k \in [K]} \sum_{l \in [L]\setminus\{1\}} O\left(\frac{\log(LT/\delta)}{\mu_{k,1} - \mu_{k,l}}\right). \tag{20}$$

Proof See Appendix C.3. ■

We provide simulation results of an instance of the matching problem showing that Algorithm 1 converges to the optimal solution in Appendix D.

5. Model Generalizations

5.1. Generalization to the Time-Varying Case

Two key properties are necessary to ensure Algorithm 1's convergence to optimality. First, the underlying statistics must be time-invariant to allow a stochastic MAB solution. Second, A2 must receive A1's choice of k_t (or exactly reproduce A1's computation); otherwise, one can still use UCB-like methods on the problem, but convergence to optimality is not guaranteed. Consider a generalization of the matching problem, where A1 observes imperfectly as well with $\mathbb{P}(O_t^1 = S_t) = \alpha$ and $\mathbb{P}(O_t^1 = -S_t) = 1 - \alpha$. Then when A1 does not receive the reward, i.e. $Y_t = 0$, it does not know whether the mistake was made by itself, or A2, or both of them; same for A2. In this case they cannot deduce S_t. For ease of presentation, we assume the environment will announce the current state at the end of a time step if they have failed to match two times in a row, so that at any time step t either S_{t-1} or S_{t-2} is CI, and we enforce $U_t = 0$ (by assigning $c > \frac{1}{1-\lambda}$ for example), i.e., no communication is allowed. Let $\bar{\alpha} = 1 - \alpha$, $\bar{\beta} = 1 - \beta$, and $\bar{\epsilon} = 1 - \epsilon$.

Now consider when the model is known at time step t. Suppose S_{t-1} is CI, so that $\mathbb{P}(S_t = \bar{S}_t) = \bar{\epsilon}$; the information state is Π_0 (we can omit the state subscript due to symmetry). Also denote the optimal policy in (18) as $g^{\text{ii}*}_{\epsilon,\beta}(\bar{S}_t, O_t)$ (note that (18) does not use the information of $(S_{t-1}, Y_{t-1}$ apart from $\bar{S}_t$). From the argument of Appendix B.3, they always want to maximize $\mathbb{P}(Y_t = 1)$, so that the optimal policies in this time step should be $g^{\text{ii}*}_{\epsilon,\alpha}(\bar{S}_t, O^1_t)$ and $g^{\text{ii}*}_{\epsilon,\beta}(\bar{S}_t, O^2_t)$ for A1 and A2, respectively, as $\mathbb{P}(Y_t = 1)$ will be maximized if both $\mathbb{P}(S_t = A^1_t)$ and $\mathbb{P}(S_t = A^2_t)$ are maximized. Depending on the regime $(\alpha, \beta, \epsilon)$ lies in, there are 16 possibilities; we assume the case of $\epsilon < \alpha, \beta$ and $\bar{\epsilon} < \alpha, \beta$ so that both agents always follow their observations, for ease of presentation. Now if they fail to match, the information state will grow to Π_1 at $t+1$, where A1 (and A2 symmetrically) can calculate the prior of the state using Bayes rule according to its private observation:

$$\begin{aligned}\mathbb{P}(S_{t+1} = \bar{S}_t | O^1_t = \bar{S}_t, Y_t = 0) &= \frac{\bar{\epsilon}\alpha\bar{\beta}\epsilon + \epsilon\bar{\alpha}\bar{\epsilon}}{\bar{\epsilon}\alpha\bar{\beta} + \epsilon\bar{\alpha}} := 1 - \epsilon'_1(O^1_t = \bar{S}_t), \\ \mathbb{P}(S_{t+1} = \bar{S}_t | O^1_t = -\bar{S}_t, Y_t = 0) &= \frac{\epsilon\alpha\bar{\beta}\bar{\epsilon} + \bar{\epsilon}\bar{\alpha}\epsilon}{\epsilon\alpha\bar{\beta} + \bar{\epsilon}\bar{\alpha}} := 1 - \epsilon'_1(O^1_t = -\bar{S}_t).\end{aligned} \tag{21}$$

Define $\epsilon'_2(O^2_t)$ similarly (by switching α and β in (21)). Then in Π_1 (at $t+1$) their optimal policies will become $g^{\text{ii}*}_{\epsilon'_1(O^1_t),\alpha}(\bar{S}_t, O^1_{t+1})$ and $g^{\text{ii}*}_{\epsilon'_2(O^2_t),\beta}(\bar{S}_t, O^2_{t+1})$.

A1's prescription functions are as follows. In Π_0, A1's optimal prescription, $g^{\text{ii}*}_{\epsilon,\alpha}(\bar{S}_t, O^1_t)$, specify whether it should follow or flip the observation for both cases of $O^1_t = \bar{S}_t$ and $O^1_t = -\bar{S}_t$. To shorten the notations, set $\bar{O} = \mathbf{1}\{O = \bar{S}\}$ and $\bar{A} = \mathbf{1}\{A = O\}$ to manifest the effective dimension of interest. Then A1's optimal prescription in state Π_0 should be a function $\gamma^{\text{i}}_0 \in \Gamma^{\text{i}}_0 := \Omega(\Omega(\bar{O}^1_t) \to \Omega(\bar{A}^1_t))$. Similarly, A1's optimal prescription in state Π_1, $g^{\text{ii}*}_{\epsilon'_1(O^1_t),\alpha}(\bar{S}_t, O^1_{t+1})$, should be a function $\gamma^{\text{i}}_1 \in \Gamma^{\text{i}}_1 := \Omega(\Omega(\bar{O}^1_t) \times \Omega(\bar{O}^1_{t+1}) \to \Omega(\bar{A}^1_{t+1}))$. Note that the two Γ's do not have subscript t due to Proposition 4.

Consider this generalization when the model (parameters) is unknown. A naive application of bandit methods is to treat A1's policy in Π_0 as two MAB problems, each has two arms (just as A2's situation in the second layer of Algorithm 1), and A1's policy in Π_0 as four MAB problems, each has two arms; same for A2. Both key properties for Algorithm 1 to work are violated. There are two states Π_0 and Π_1, hence the environment is time-varying; one cannot treat the problem as six MABs with two arms either, since the choice of γ^{i}_0 affects the probability of entering Π_1, and maximizing the reward in Π_0 may not aligned with maximizing $\mathbb{P}(\text{enter } \Pi_1)V(\Pi_1)$. In addition, when they fail to match, they do not receive each other's actions, so that the information structure in Protocol 1 does not hold. However, the CI approach illuminates a path using Q-learning so that convergence to optimality is guaranteed. In particular, as the DP equation in Proposition 4 suggests, we treat $\{\Pi_0, \Pi_1\}$ as the state space, $\Gamma^{\text{i}}_0 \times \Gamma^{\text{ii}}_0$ as the action space of Π_0, and $\Gamma^{\text{i}}_1 \times \Gamma^{\text{ii}}_1$ as the action space of Π_1, then the problem can be solved with Q-learning when the model is unknown. Since the learning is done by the coordinator using only CI, the non-stationarity issue in MARL problem is alleviated. Essentially, this scheme establishes a common ground between the two agents, so that any learning done in one agent can be computationally reproduced by the other.

5.2. Generalization to the Asymmetric Case

When the transition and observation probabilities do not have the symmetric structure as in Section 2, A1 does not necessarily match the state in the optimal policy. The following is one such example. For the transition probabilities, we have

- $\mathbb{P}(S_t = +1|S_{t-1} = +1, Y_{t-1} = 1) = \mathbb{P}(S_t = +1|S_{t-1} = -1, Y_{t-1} = 0) = 0.99$,
- $\mathbb{P}(S_t = -1|S_{t-1} = +1, Y_{t-1} = 1) = \mathbb{P}(S_t = -1|S_{t-1} = -1, Y_{t-1} = 0) = 0.01$,
- $\mathbb{P}(S_t = +1|S_{t-1} = -1, Y_{t-1} = 1) = \mathbb{P}(S_t = +1|S_{t-1} = +1, Y_{t-1} = 0) = 0.49$, and
- $\mathbb{P}(S_t = -1|S_{t-1} = -1, Y_{t-1} = 1) = \mathbb{P}(S_t = -1|S_{t-1} = +1, Y_{t-1} = 0) = 0.51$.

For the observation probabilities, we have $\mathbb{P}(O_t = +1|S_t = +1) = \mathbb{P}(O_t = +1|S_t = -1) = \beta$ and $\mathbb{P}(O_t = -1|S_t = +1) = \mathbb{P}(O_t = -1|S_t = -1) = 1 - \beta$, where $\beta \in (0, 1)$; the observation is basically useless for A2, and A2 will guess the state solely from the prior. We again forbid communication by assigning $c > \frac{1}{1-\lambda}$, where $\lambda = 0.9$.

Now suppose A1 always matches the state. Then A2 disregards observations, and always guesses $S_t = +1$ if (Y_{t-1}, S_{t-1}) is equal to $(1, +1)$ or $(0, -1)$, and always guesses $S_t = -1$ if (Y_{t-1}, S_{t-1}) is equal to $(1, -1)$ or $(0, +1)$. This means the two sets of information states (where the tuples are in the form of (Y_{t-1}, S_{t-1})) $\{(1, +1), (0, -1)\}$ and $\{(1, -1), (0, +1)\}$ are not commutative under the policy. For A1, at each time step there are eight possible inputs of (Y_{t-1}, S_{t-1}, S_t). For the four inputs that have (Y_{t-1}, S_{t-1}) equal to $(1, +1)$ or $(0, -1)$, the values are close to $1/(1 - 0.9) = 10$, while the four inputs that have (Y_{t-1}, S_{t-1}) equal to $(1, -1)$ or $(0, +1)$ have values close to $0.5/(1 - 0.9) = 5$. Then when $(Y_{t-1}, S_{t-1}, S_t) = (1, -1, -1)$, instead of getting 5 (more precisely 5.59), A1 could gain by not matching the state so that in the next time step they have $(Y_{t-1}, S_{t-1}) = (0, -1)$, obtaining roughly $\lambda \cdot 10 = 9$ (more precisely 8.91). This implies always matching is not optimal, which destroys the good properties of Corollary 6 and Remark 7. Characterizing the optimal policy in this case with the corresponding reinforcement learning method are left as future work.

5.3. Generalization to More Than Two States

From the last Subsection, it is easy to imagine that with more than two states and no particular structure enforced on the probabilities, one could also find examples such that A1 always matching is not optimal. Here is a concise one with three states. Suppose $\mathcal{S} = \{I, J, K\}$ and $\mathcal{O} = \{i, j, k\}$. If $Y_{t-1} = 1$, then $\mathbb{P}(S_t = I|S_{t-1} = I) = \mathbb{P}(S_t = J|S_{t-1} = I) = \mathbb{P}(S_t = I|S_{t-1} = J) = \mathbb{P}(S_t = J|S_{t-1} = J) = 0.5$, $\mathbb{P}(S_t = I|S_{t-1} = K) = \mathbb{P}(S_t = J|S_{t-1} = K) = 0.1$, and $\mathbb{P}(S_t = K|S_{t-1} = K) = 0.8$; if $Y_{t-1} = 0$, then all states transit to $S_t = K$. For the observation probabilities, we have $\mathbb{P}(O_t = i|S_t = I) = \mathbb{P}(O_t = j|S_t = I) = \mathbb{P}(O_t = i|S_t = J) = \mathbb{P}(O_t = j|S_t = J) = 0.5$ and $\mathbb{P}(O_t = k|S_t = K) = 1$. Let $\lambda = 0.9$. The two states I and J are the "bad states" where A2 has unclear observation and does no better than randomly guessing between them, while in state K A2 observes perfectly. In I or J, instead of getting a 0.5 instantaneous reward, A1 would rather not matching the state and bringing them back to state K, so that they will earn 0.9 in the next step even with discounting.

Again, without Corollary 6 and Remark 7 it is unclear how to characterize the optimal policy when the model is known, let alone when it is unknown. Our Q-learning method works in Subsection 5.1 because we assume that tge system will announce the state if they fail to match two times.

As we learn from Proposition 4, this limits the length of AOH and CI required for optimal decision to two, so that the state space and the action space in the Q-learning method are finite. For general MARL problem, the length could go to infinity, and one needs to approximate the information state by discarding part of the history to construct an algorithm with reasonable complexity. This will also be left as future work.

6. Conclusions

In this paper we studied the applicability of the common information approach to solve multi-agent cooperative reinforcement learning problems. This we did by studying a decentralized matching problem between two agents. The common information approach allowed us to develop optimal policies when the model was known, and this was then used to devise bandit-based learning algorithms to achieve close to optimal regret.

An important part of our paper was to highlight the critical role played by inter-agent communications in increasing the common information. While we did not explore this aspect, in the reinforcement learning context, communications can have the added benefit of reducing the complexity of the solutions by collapsing the information retained at agents. In future work, we plan on developing these ideas in more complex multi-agent settings.

References

R. Agrawal. Sample mean based index policies by o(log n) regret for the multi-armed bandit problem. *Advances in Applied Probability*, 27(4):1054–1078, 1995.

P. Auer, N. Cesa-Bianchi, and P. Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine learning*, 47(2-3):235–256, 2002.

D. S. Bernstein, R. Givan, N. Immerman, and S. Zilberstein. The complexity of decentralized control of markov decision processes. *Mathematics of operations research*, 27(4):819–840, 2002.

L. Buşoniu, R. Babuška, and B. De Schutter. A comprehensive survey of multiagent reinforcement learning. *IEEE Trans. on Systems, Man, and Cybernetics, Part C (Applications and Reviews)*, 38 (2):156–172, Mar. 2008.

C. Claus and C. Boutilier. The dynamics of reinforcement learning in cooperative multiagent system. In *AAAI Conference on Artificial Intelligence*, 1998.

C. A. S. de Witt, J. N. Foerster, G. Farquhar, P. H. S. Torr, W. Böhmer, and S. Whiteson. Multi-agent common knowledge reinforcement learning. In *Advances in Neural Information Processing Systems (NIPS)*, 2019.

J. N. Foerster, H. F. Song, E. Hughes, N. Burch, I. Dunning, S. Whiteson, M. Botvinick, and M. Bowling. Bayesian action decoder for deep multi-agent reinforcement learning. In *Proc. 36th International Conference on Machine Learning (ICML 2019)*, 2019.

E. A. Hansen, D. S. Bernstein, and S. Zilberstein. Dynamic programming for partially observable stochastic games. In *AAAI Conference on Artificial Intelligence*, 2004.

P. Hernandez-Leal, M. Kaisers, T. Baarslag, and E. Munoz de Cote. A survey of learning in multiagent environments: Dealing with non-stationarity. Available at https://arxiv.org/abs/1707.09183, 2019.

T. Başar K. Zhang, Z. Yang. Multi-agent reinforcement learning: A selective overview of theories and algorithms. Available at https://arxiv.org/abs/1911.10635, 2019.

S. Kar, J. M. F. Moura, and H. V. Poor. Qd-learning a collaborative distributed strategy for multi-agent reinforcement learning through concensus + innovations. *IEEE Trans. on Signal Processing*, 61(7):1848–1862, Apr. 2013.

A. Lerer, H. Hu, J. Foerster, and N. Brown. Improving policies via search in cooperative partially observable games. Available at https://arxiv.org/abs/1912.02318, 2019.

P. Naghizadeh, M. Gorlatovay, A. A. Lanz, and M. Chiang. Hurts to be too early: Benefits and drawbacks of communication in multi-agent learning. In *2019 IEEE INFOCOM*, 2019.

A. Nayyar, T. Başar, D. Teneketzis, and V. V. Veeravalli. Optimal strategies for communication and remote estimation with an energy harvesting sensor. *IEEE Trans. on Automatic Control*, 58(9): 2246–2260, Sep. 2013a.

A. Nayyar, A. Mahajan, and D. Teneketzis. Decentralized stochastic control with partial history sharing: A common information approach. *IEEE Trans. on Automatic Control*, 58(7):1644–1658, July 2013b.

A. Nayyar, A. Mahajan, and D. Teneketzis. The common-information approach to decentralized stochastic control. In *Information and Control in Networks*, pages 123–156. Springer, 2014.

A. Nedić and A. Ozdaglar. Distributed subgradient methods for multiagent optimization. *IEEE Trans. on Automatic Control*, 54(1):48–61, 2009.

Y. Ouyang, H. Tavafoghi, and D. Teneketzis. Dynamic games with asymmetric information: Common information based perfect bayesian equilibria and sequential decomposition. *IEEE Trans. on Automatic Control*, 62(1):222–237, Jan. 2017.

R. S. Sutton and A. G. Barto. *Reinforcement Learning: An Introduction*. MIT Press, Cambridge, 2018.

H. Tavafoghi, Y. Ouyang, and D. Teneketzis. A sufficient information approach to decentralized decision making. In *2018 IEEE 57th Annual Conference on Decision and Control (CDC)*, 2018.

H. S. Witsenhausen. A standard form for sequential stochastic control. *Mathematical Systems Theory*, 7(1):5–11, 1973.

M. Wunder, M. Kaisers, J. R. Yaros, and M. L. Littman. Using iterated reasoning to predict opponent strategies. In *Proceedings of 10th International Conference on Autonomous Agents and Multiagent Systems*, 2011.

M. Wunder, J. R. Yaros, M. Kaisers, and M. L. Littman. A framework for modeling population strategies by depth of reasoning. In *Proceedings of 11th International Conference on Autonomous Agents and Multiagent Systems*, 2012.

K. Zhang, Z. Yang, H. Liu, T. Zhang, and T. Başar. Fully decentralized multi-agent reinforcement learning with networked agents. In *Proc. 35th International Conference on Machine Learning (ICML 2018)*, 2018.

Appendix A. Proofs of the Structural Results

A.1. Proof of Lemma 1

We only prove the case for the first sub-step. The proof of the second sub-step is the same. Notice that whenever $Z_t \neq 0$ or $Y_t \neq 0$, S_t is in the CI – if $Z_t = U_t S_t \neq 0$, it means $U_t = 1$ and $Z_t = S_t$, and since Z_t is CI, so is S_t; on the other hand, if $Y_t = \mathbf{1}\{S_t = A_t^1 = A_t^2\} = 1 \neq 0$, then both agents can deduce S_t from their private information, namely past actions A_t^1 and A_t^2.

Now the CI is $(Z_{1:t-1}, Y_{1:t-1}, S_{\bar{t}^{\mathrm{i}}})$. Note that since we are considering an MDP, given a strategy over time (that is a $g_{1:\infty}$), $S_{\bar{t}^{\mathrm{i}}}$ will be the sufficient statistics of all variables prior to $\bar{t}^{\mathrm{i}}$ when it comes to the statistics of any variables after $\bar{t}^{\mathrm{i}}$. Consider an optimal prescription $\gamma_t^{\mathrm{i}*} \in \Omega(\Omega(S_{1:t}) \times \Omega(A^1_{1:t-1}) \to \Omega(U_t))$ and two admissible histories $m_1, m_2 \in \Omega(S_{1:\bar{t}^{\mathrm{i}}-1}) \times \Omega(A^1_{1:\bar{t}^{\mathrm{i}}-1})$ and $m_3 \in \Omega(S_{\bar{t}^{\mathrm{i}}:t}) \times \Omega(A^1_{\bar{t}^{\mathrm{i}}:t-1})$ so that both (m_1, m_3) and (m_2, m_3) are consistent with $(Z_{1:t-1}, Y_{1:t-1}, S_{\bar{t}^{\mathrm{i}}})$ (which means the probabilities of the two histories are positive given the CI). Denote the value of the first equation of (5) before taking the supremum as $V_t^{\mathrm{i}}(\pi_t^{\mathrm{i}}, \gamma_t^{\mathrm{i}})$. Then we claim it must be that

$$V_t^{\mathrm{i}}(\pi_t^{\mathrm{i}}, \gamma_t^{\mathrm{i}*}(m_1, m_3), M_{1:t} = (m_1, m_3)) = V_t^{\mathrm{i}}(\pi_t^{\mathrm{i}}, \gamma_t^{\mathrm{i}*}(m_2, m_3), M_{1:t} = (m_2, m_3)). \tag{22}$$

We use proof by contradiction. Suppose the left hand side is larger. Then we can construct a new prescription $\gamma_t^{\mathrm{i}**}$ such that it is the same as $\gamma_t^{\mathrm{i}*}$ for all histories except that $\gamma_t^{\mathrm{i}**}(m_2, m_3) = \gamma_t^{\mathrm{i}*}(m_1, m_3)$. Then when (m_2, m_3) happens, the future dynamics of adopting $\gamma_t^{\mathrm{i}**}$ will be exactly the same as when (m_1, m_3) happens and $\gamma_t^{\mathrm{i}*}$ is adopted since $S_{\bar{t}^{\mathrm{i}}}$ is known; and since (m_2, m_3) happens with a positive probability, we have $V_t^{\mathrm{i}}(\pi_t^{\mathrm{i}}, \gamma_t^{\mathrm{i}**}) > V_t^{\mathrm{i}}(\pi_t^{\mathrm{i}}, \gamma_t^{\mathrm{i}*})$, a contradiction. We can then construct an optimal strategy $\gamma_t^{\mathrm{i}***} \in \Omega(\Omega(S_{\bar{t}^{\mathrm{i}}+1:t}) \times \Omega(A^1_{\bar{t}^{\mathrm{i}}:t-1}) \to \Omega(U_t))$ from $\gamma_t^{\mathrm{i}*}$ such that for any $m' \in \Omega(S_{\bar{t}^{\mathrm{i}}+1:t}) \times \Omega(A^1_{\bar{t}^{\mathrm{i}}:t-1}) \to \Omega(U_t)$, we choose arbitrary consistent $m \in \Omega(S_{1:\bar{t}^{\mathrm{i}}-1}) \times \Omega(A^1_{1:\bar{t}^{\mathrm{i}}-1})$ such that $\gamma_t^{\mathrm{i}***}(m') = \gamma_t^{\mathrm{i}*}(m, m')$, and from (22) we have $V_t^{\mathrm{i}}(\pi_t^{\mathrm{i}}, \gamma_t^{\mathrm{i}*}) = V_t^{\mathrm{i}}(\pi_t^{\mathrm{i}}, \gamma_t^{\mathrm{i}***})$, which implies $\gamma_t^{\mathrm{i}***}$ is also an optimal prescription.

A.2. Proof of Lemma 3

Again we only prove the case for the first sub-step. Note that given $(Z_{1:t-1}, Y_{1:t-1})$ and the fact that $\bar{t}^{\mathrm{i}}$ is well-defined, $S_{\bar{t}^{\mathrm{i}}}$ is also in the CI, so that

$$\begin{aligned}
&\Pi_t^{\mathrm{i}}(m, m') \\
&= \mathbb{P}^{g_{1:t-1}}(M_{1:t} = (m, m') | Z_{1:t-1}, Y_{1:t-1}) \\
&= \mathbb{P}^{g_{1:t-1}}(M_{1:t} = (m, m') | S_{\bar{t}^{\mathrm{i}}}, Z_{1:t-1}, Y_{1:t-1}) \\
&= \mathbb{P}^{g_{1:\bar{t}^{\mathrm{i}}-1}}(M_{1:\bar{t}^{\mathrm{i}}} = m | S_{\bar{t}^{\mathrm{i}}}, Z_{1:t-1}, Y_{1:t-1}) \cdot \mathbb{P}^{g_{\bar{t}^{\mathrm{i}}:t-1}}(M_{\bar{t}^{\mathrm{i}}+1:t} = m' | M_{1:\bar{t}^{\mathrm{i}}} = m, S_{\bar{t}^{\mathrm{i}}}, Z_{1:t-1}, Y_{1:t-1}) \\
&= \mathbb{P}^{g_{1:\bar{t}^{\mathrm{i}}-1}}(M_{1:\bar{t}^{\mathrm{i}}} = m | Z_{1:\bar{t}^{\mathrm{i}}-1}, Y_{1:\bar{t}^{\mathrm{i}}-1}) \cdot \mathbb{P}^{g_{\bar{t}^{\mathrm{i}}:t-1}}(M_{\bar{t}^{\mathrm{i}}+1:t} = m' | S_{\bar{t}^{\mathrm{i}}}, Z_{\bar{t}^{\mathrm{i}}:t-1}, Y_{\bar{t}^{\mathrm{i}}:t-1}) \\
&= \Pi_{\bar{t}^{\mathrm{i}}}^{\mathrm{i}}(m) \cdot \Pi_{\bar{t}^{\mathrm{i}}+1:t}^{\mathrm{i}}(m').
\end{aligned}$$

The penultimate equality follows from two crucial facts. First, $S_{\bar{t}^{\mathrm{i}}}$ is the sufficient statistics for the previous variables given the strategies. Second, using the strategy space reduction from Lemma 1, the strategies also do not depend on $M_{1:\bar{t}^{\mathrm{i}}} = m$; hence, it can be removed from the conditioning.

A.3. An Intermediate Proposition 11 and Its Proof

Proposition 11 *Consider time step t. Given the CI $(Z_{1:t-1}, Y_{1:t-1})$, the DP equation for the first sub-step can be written as*

$$V_t^{\mathrm{i}}(\pi^{\mathrm{i}}_{\bar{t}^{\mathrm{i}}+1:t}) = \sup_{\gamma_t^{\mathrm{i}}} \mathbb{E}\left[-c\gamma_t^{\mathrm{i}}(M_{\bar{t}^{\mathrm{i}}+1:t}) + V_t^{\mathrm{ii}}(\eta^{\mathrm{i}}_{\bar{t}^{\mathrm{i}},t}(\Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{i}}+1:t}, \gamma_t^{\mathrm{i}}, Z_t)) \middle| \Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{i}}+1:t} = \pi^{\mathrm{i}}_{\bar{t}^{\mathrm{i}}+1:t}\right], \tag{23}$$

where $\gamma_t^{\mathrm{i}} \in \Omega(\Omega(S_{\bar{t}^{\mathrm{i}}+1:t}) \times \Omega(A^1_{\bar{t}^{\mathrm{i}}:t-1}) \to \Omega(U_t))$ and

$$\Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{i}}+1:t}(m) = \eta^{\mathrm{i}}_{\bar{t}^{\mathrm{i}},t}(\Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{i}}+1:t}, \gamma_t^{\mathrm{i}}, Z_t)(m) = \frac{\Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{i}}+1:t}(m)\mathbb{P}(Z_t|M_{\bar{t}^{\mathrm{i}}+1:t} = m, \gamma_t^{\mathrm{i}})}{\sum_{m'} \Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{i}}+1:t}(m')\mathbb{P}(Z_t|M_{\bar{t}^{\mathrm{i}}+1:t} = m', \gamma_t^{\mathrm{i}})}, \tag{24}$$

where $m, m' \in \Omega(M_{\bar{t}^{\mathrm{i}}+1:t})$. Similarly, the DP equation for the second sub-step can be written as

$$V_t^{\mathrm{ii}}(\pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t}) = \sup_{\gamma_t^{\mathrm{ii}}} \mathbb{E}\left[Y_t(S_t, \gamma_t^{\mathrm{ii}}(M_{\bar{t}^{\mathrm{ii}}+1:t})) + \lambda V^{\mathrm{i}}_{t+1}(\eta^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}},t}(\Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t}, \gamma_t^{\mathrm{ii}}, Y_t)) \middle| \Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t} = \pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t}\right], \tag{25}$$

where $\gamma_t^{\mathrm{ii}} \in \Omega(\Omega(M_{\bar{t}^{\mathrm{ii}}+1:t}) \to \Omega(A_t))$ and

$$\begin{aligned}\Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{ii}}+1:t+1}(m, m') &= \eta^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}},t}(\Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t}, \gamma_t^{\mathrm{ii}}, Y_t)(m, m') \\ &= \frac{\Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t}(m)\mathbb{P}(M_{t+1} = m', Y_t|M_{\bar{t}^{\mathrm{ii}}+1:t} = m, \gamma_t^{\mathrm{ii}})}{\sum_{\bar{m},\bar{m}'} \Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t}(\bar{m})\mathbb{P}(M_{t+1} = \bar{m}', Y_t|M_{\bar{t}^{\mathrm{ii}}+1:t} = \bar{m}, \gamma_t^{\mathrm{ii}})},\end{aligned} \tag{26}$$

where $m, \bar{m} \in \Omega(M_{\bar{t}^{\mathrm{ii}}+1:t})$ and $m', \bar{m}' \in \Omega(M_{t+1})$. If $Z_t = 0$, then $\bar{t}^{\mathrm{ii}} = \bar{t}^{\mathrm{i}} \le t-1$ and $\Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t}(m) = \Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{i}}+1:t}(m)$; otherwise, $\bar{t}^{\mathrm{ii}} = t > t-1 \ge \bar{t}^{\mathrm{i}}$, then we let $\Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t}(m)$ to be the null information state Π^{ii}_0, *which is a distribution on nothing*[4]. *Similarly, if $Y_t = 0$, then $\overline{t+1}^{\mathrm{i}} = \bar{t}^{\mathrm{ii}} \le t$ and $\Pi^{\mathrm{i}}_{\overline{t+1}^{\mathrm{i}}+1:t+1}(m) = \Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{ii}}+1:t+1}(m)$; otherwise, $\overline{t+1}^{\mathrm{i}} = t \ge \bar{t}^{\mathrm{ii}}$ and we split $\Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{ii}}+1:t+1}(m, m') = \Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{ii}}+1:\overline{t+1}^{\mathrm{i}}}(m) \cdot \Pi^{\mathrm{i}}_{\overline{t+1}^{\mathrm{i}}+1:t+1}(m')$ and let the second half be the new $\Pi^{\mathrm{i}}_{\overline{t+1}^{\mathrm{i}}+1:t+1}(m')$, throwing away the first half*[5].

Proof First, from Lemma 1 we can restrict attention to $\gamma_t^{\mathrm{i}} \in \Omega(\Omega(S_{\bar{t}^{\mathrm{i}}+1:t}) \times \Omega(A^1_{\bar{t}^{\mathrm{i}}:t-1}) \to \Omega(U_t))$. Notice that

$$\begin{aligned}&\Pi_t^{\mathrm{i}}(m, m') \\ &= \mathbb{P}^{g_{1:\bar{t}^{\mathrm{i}}-1}, g^{\mathrm{i}}_{\bar{t}^{\mathrm{i}}}}(M_{1:\bar{t}^{\mathrm{i}}} = m|Z_{1:\bar{t}^{\mathrm{i}}-1}, Y_{1:\bar{t}^{\mathrm{i}}-1}) \cdot \mathbb{P}^{g_{\bar{t}^{\mathrm{i}}:t-1}}(M_{\bar{t}^{\mathrm{i}}+1:t} = m'|S_{\bar{t}^{\mathrm{i}}}, Z_{\bar{t}^{\mathrm{i}}:t}, Y_{\bar{t}^{\mathrm{i}}:t-1}) \\ &= \Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{i}}}(m) \cdot \Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{i}}+1:t}(m'),\end{aligned}$$

as $\bar{t}^{\mathrm{i}} \le t-1$. This can be seen as a variant of (10) from Lemma 3. We can use this and (11) from Lemma 3 and split all future information states $\Pi^{\mathrm{i}}_{t'}(m, m') = \Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{i}}}(m) \cdot \Pi^{\mathrm{i}}_{\bar{t}^{\mathrm{i}}+1:t'}(m')$ and $\Pi^{\mathrm{ii}}_{t'}(m, m') = \Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{i}}}(m) \cdot \Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{i}}+1:t'}(m')$. The first part of the states $\Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{i}}}(m)$ is henceforth irrelevant and can be thrown away. The key is the update rule in (6) for the information state can be done

4. Intuitively, we just split $\Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{i}}+1:t}(m, m') = \Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{i}}+1:\bar{t}^{\mathrm{ii}}}(m) \cdot \Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t}(m')$ and throw away the first half and let the second half be $\Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t}(m')$. Note that $\Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{ii}}+1:t}(m') = \Pi^{\mathrm{ii}}_{t+1:t}(m')$, so we actually throw away the whole $\Pi^{\mathrm{ii}}_{\bar{t}^{\mathrm{i}}+1:t}(m)$.

5. The first half is *null* (which means there is no first half) if $\overline{t+1}^{\mathrm{i}} = t = \bar{t}^{\mathrm{ii}}$.

without $\Pi^{\text{ii}}_{\bar{t}^{\text{i}}}(m)$. Consider the right hand side of (6). The term $\mathbb{P}(Z_t|M_{1:t} = (m, m'), \gamma^{\text{i}}_t)$ is actually $\mathbb{P}(Z_t|M_{\bar{t}+1:t} = m', \gamma^{\text{i}}_t)$ since $\gamma^{\text{i}}_t \in \Omega(\Omega(S_{\bar{t}^{\text{i}}+1:t}) \times \Omega(A^1_{\bar{t}^{\text{i}}:t-1}) \to \Omega(U_t))$. Hence, it can be rewritten as

$$\begin{aligned}\Pi^{\text{ii}}_t(m, m') &= \Pi^{\text{ii}}_{\bar{t}^{\text{i}}}(m) \cdot \Pi^{\text{ii}}_{\bar{t}^{\text{i}}+1:t}(m') \\ &= \frac{\Pi^{\text{i}}_t(m, m')\mathbb{P}(Z_t|M_{1:t} = (m, m'), \gamma^{\text{i}}_t)}{\sum_{\bar{m},\bar{m}'} \Pi^{\text{i}}_t(\bar{m}, \bar{m}')\mathbb{P}(Z_t|M_{1:t} = (\bar{m}, \bar{m}'), \gamma^{\text{i}}_t)} \\ &= \frac{\Pi^{\text{ii}}_{\bar{t}^{\text{i}}}(m) \cdot \Pi^{\text{i}}_{\bar{t}^{\text{i}}+1:t}(m') \cdot \mathbb{P}(Z_t|M_{\bar{t}+1:t} = m', \gamma^{\text{i}}_t)}{\sum_{\bar{m},\bar{m}'} \Pi^{\text{ii}}_{\bar{t}^{\text{i}}}(\bar{m}) \cdot \Pi^{\text{i}}_{\bar{t}^{\text{i}}+1:t}(\bar{m}') \cdot \mathbb{P}(Z_t|M_{\bar{t}+1:t} = \bar{m}', \gamma^{\text{i}}_t)} \\ &= \Pi^{\text{ii}}_{\bar{t}^{\text{i}}}(m) \cdot \frac{\Pi^{\text{i}}_{\bar{t}^{\text{i}}+1:t}(m')\mathbb{P}(Z_t|M_{\bar{t}+1:t} = m', \gamma^{\text{i}}_t)}{\sum_{\bar{m}'} \Pi^{\text{i}}_{\bar{t}^{\text{i}}+1:t}(\bar{m}') \cdot \mathbb{P}(Z_t|M_{\bar{t}+1:t} = \bar{m}', \gamma^{\text{i}}_t)},\end{aligned}$$

which simplifies to (24). ■

A.4. Proof of Proposition 4

We need the stationarity of the model here – in every time step, the transition kernel, observation binary symmetric channel (BSC), as well as the reward/cost function are exactly the same. Hence, the dynamic equations (23) and (25) in Proposition 11 do not depend on the values of $\bar{t}^{\text{i}}$ and $\bar{t}^{\text{ii}}$. Consider $V^{\text{i}}_t(\pi^{\text{i}}_{\bar{t}^{\text{i}}+1:t})$ in Proposition 11 and an optimal prescription $\gamma^{\text{i}*}_t$ that achieves it. If we have some $t' \neq t$, the CI so that $t' - \bar{t}' = t - \bar{t}$, and the same belief state $\pi^{\text{i}}_{\bar{t}'^{\text{i}}+1:t'} = \pi^{\text{i}}_{\bar{t}^{\text{i}}+1:t}$. Then $\gamma^{\text{i}*}_t$ will work as the optimal prescription here, since it will lead to the same dynamics as the old t problem, and one could not do any better because any improvement in the t' problem works as an improvement in the old t problem. We can see that the DP problems are time-invariant, and the DP equations only depend on the difference between $\bar{t}^{\text{i}}$ and t and between $\bar{t}^{\text{ii}}$ and t.

Appendix B. Proofs of Characterizing the Optimal Policy

B.1. Proof of Theorem 5

A complete policy is of the form $g_{1:\infty} = (g^{\text{i}}_{1:\infty}, g^{\text{ii},1}_{1:\infty}, g^{\text{ii},2}_{1:\infty})$. Denote the policy that A1 always matches the state for all t as $g^{\text{ii},1\sharp}_{1:\infty}$. For the remaining parts of the complete policy, we can find an optimal one, denoted as $(g^{\text{i}\sharp}_{1:\infty}, g^{\text{ii},2\sharp}_{1:\infty})$. Note that with $g^{\sharp}_{1:\infty} = (g^{\text{i}\sharp}_{1:\infty}, g^{\text{ii},1\sharp}_{1:\infty}, g^{\text{ii},2\sharp}_{1:\infty})$, we have Corollary 6 and Remark 7, that is, the previous state S_{t-1} is CI and that the policy is stationary, which means there is a $g^{\sharp}_{1:\infty}$ consisting of playing the same policy $g^{\sharp} = (g^{\text{i}\sharp}, g^{\text{ii},1\sharp}, g^{\text{ii},2\sharp})$ at each time instant [6].

The following proof will adopt the indexing of 0 as the current time step. We merge the second sub-step into the first sub-step so that we consider V^{i} and π^{i} but drop their superscripts. We consider two policies. The first one is $g^{opt} = (g^{\text{i}\sharp}, g^{\text{ii},1\sharp}, g^{\text{ii},2\sharp}, g^{\sharp}_{1:\infty})$, i.e., it plays $g^{\sharp}$ at the current time step and all the time steps afterwards (so it is just a shifted version of $g^{\sharp}_{1:\infty}$). The second one is $g^{dev} = (g^{\text{i}}_0, g^{\text{ii},1\flat}, g^{\text{ii},2}_0, g^{\text{i}}_1, g^{\text{ii},1\sharp}, g^{\text{ii},2}_1, g^{\sharp}_{1:\infty})$ for arbitrary g^{i}_0, $g^{\text{ii},2}_0$, g^{i}_1, and $g^{\text{ii},2}_1$, where $g^{\text{ii},1\flat}$ is the policy that A1 does not match the state at the current time step. Denote the value function for an information state π under a complete policy g as $V(\pi, g)$, so that $V(\pi) = \sup_g V(\pi, g)$. We show that for arbitrary π in the first sub-step, $V(\pi, g^{opt}) > V(\pi, g^{dev})$.

6. This does not mean the policy is *myopic* – the policy could still take later stages into account, it just does not change over time.

Recall that by Proposition 4, the information state in the first sub-step is of the form $\pi_{\tau,s,v}$ where $\tau \geq 0$, $s \in \{+1,-1\}$, and $v \in \{y, z\}$ (we drop the $^{\mathrm{i}}$ superscript). Then by Corollary 6, whenever $g^\sharp$ is adopted in the previous time step, we have $\tau = 1$ and $v = y$, which means there are only two possible information states, namely, $\pi_{1,+1,y}$ and $\pi_{1,-1,y}$ (actually four, if one further splits the cases of $y = 0$ and $y = 1$). By symmetries of the transition and observation probabilities, we must have $V(\pi_{1,+1,y}, g^\sharp_{1:\infty}) = V(\pi_{1,-1,y}, g^\sharp_{1:\infty})$. Moreover, if we denote the instantaneous reward for an information state π under a stage policy g as $R(\pi, g)$ [7], then

$$R(\pi_{1,+1,y}, g^\sharp) = R(\pi_{1,-1,y}, g^\sharp) = \xi \geq 0.5 > 0, \tag{27}$$

and

$$\begin{aligned} V(\pi_{1,+1,y}, g^\sharp_{1:\infty}) &= R(\pi_{1,+1,y}, g^\sharp) + \sum_{t=1}^{\infty} \lambda^t \left[(1-\epsilon)R(\pi_t^1, g^\sharp) + \epsilon R(\pi_t^2, g^\sharp)\right] \\ &= \xi + \frac{\lambda}{1-\lambda}\xi = \frac{1}{1-\lambda}\xi, \end{aligned} \tag{28}$$

where one of π_t^1 and π_t^2 is $\pi_{1,+1,y}$ and the other is $\pi_{1,-1,y}$.

It now becomes clear that $V(\pi, g^{opt}) = \frac{1}{1-\lambda}\xi$ for arbitrary π (we can assume $g^\sharp$ was adopted before the current time step as well). On the other hand, for arbitrary π, g^{dev} receives 0 in the current time step, not more than $\lambda\xi$ in the next time step, and then $\frac{\lambda^2}{1-\lambda}\xi$ afterwards (since A1 starts to match again in the next time step). In the next time step, they could not perform better than ξ, since whatever they do could have been done by $g^\sharp$ as well. The fact that A1 does not match in the current step will change the dynamics; however, A2's a priory knowledge of S_1 is at best ϵ and $1-\epsilon$ (on $+1$ and -1, one way or the other) due to symmetry, and it might be worse if A1 does not communicate in both time steps. Therefore,

$$V(\pi, g^{dev}) \leq 0 + \lambda\xi + \frac{\lambda^2}{1-\lambda}\xi = \frac{\lambda}{1-\lambda}\xi < V(\pi, g^{opt}). \tag{29}$$

We conclude that they cannot gain more by A1's deviation of not matching in one step, but we already assume that the other parts of the policy $g^\sharp_{1:\infty}$ are optimal under $g^{\mathrm{ii},1\sharp}_{1:\infty}$. Hence, by the Policy Improvement Theorem (Sutton and Barto, 2018), $g^\sharp_{1:\infty}$ is an optimal policy.

B.2. Proof of Corollary 6

We know from Theorem 5 that under an optimal strategy A1 always plays $A_t^1 = S_t$. At time step t, A2 already observed $Y_{t-1} = \mathbf{1}\{S_{t-1} = A_{t-1}^1 = A_{t-1}^2\} = \mathbf{1}\{S_{t-1} = A_{t-1}^2\}$, and can thus deduce S_{t-1} by $S_{t-1} = A_{t-1}^2 \cdot (-1)^{Y_{t-1}+1}$ from its private information A_{t-1}^2. Now that both agents know S_{t-1}, it is CI.

Recall the simplification results from Lemma 1 to Proposition 4 using $\bar{t}$ are basically done by exploiting the crucial fact of $S_{\bar{t}}$ being in the CI. Now that we have S_{t-1} being in the CI, Proposition 4 directly follows with $\bar{t}^{\mathrm{i}} = t-1$, and $\bar{t}^{\mathrm{ii}} = t-1$ if $U_t = 0$ and $\bar{t}^{\mathrm{ii}} = t$ if $U_t = 1$. These lead to the information state Π_1^{i} in the first sub-step, and information state Π_1^{ii} if $U_t = 0$ and Π_0^{ii} if $U_t = 1$.

7. A complete policy is that for the entire process, and a stage policy is only for a time step.

B.3. Proof of Theorem 8

We will again adopt the 0 indexing and merge the second sub-step into the first sub-step while omitting the $^{\mathrm{i}}$ superscript. As illustrated in Corollary 6, S_{-1} is part of the CI, so that any history before $t = -1$ can be dropped. Furthermore, since S_{-1} and Y_{-1} alone characterize the statistics of S_0, Z_{-1} which is also in the CI can be dropped as well. We hence write $\pi_{s,y}$ to denote the realization of the information state Π (Π_1^{i} to be precise), where $s \in \{+1, -1\}$ is the realization of S_{-1} and $y \in \{0, 1\}$ is the realization of Y_{-1}. As stated in (27), due to symmetries, all four states have the same instantaneous reward and long term value under an optimal policy, and this reward can be achieved by the same structure of policy again because of the symmetric transition structure. Specifically, for all four realizations of S_{-1} and Y_{-1}, A2's prior (or the coordinator's prior, or the prior obtained from CI) of S_0 is always ϵ on one of the state and $1 - \epsilon$ on the other:

$$S_0 = \begin{cases} S_{-1} \cdot (-1)^{Y_{-1}+1}, & \text{with probability } 1 - \epsilon, \\ S_{-1} \cdot (-1)^{Y_{-1}}, & \text{with probability } \epsilon. \end{cases} \tag{30}$$

Let us consider A1's policy $U_0 = g^{\mathrm{i}}(S_{-1}, Y_{-1}, S_0)$ first. Let $\bar{S}_0 = S_{-1} \cdot (-1)^{Y_{-1}+1}$. Then A2's prior of S_0 is $1 - \epsilon$ on $\bar{S}_0$, and ϵ on $-\bar{S}_0$. This means the case of $S_{-1} = +1$ and $Y_{-1} = 1$ will lead to the same prior as the case of $S_{-1} = -1$ and $Y_{-1} = 0$ and can share the same optimal policy. Moreover, the case of $S_{-1} = +1$ and $Y_{-1} = 0$ also shares the same prior as the case of $S_{-1} = +1$ and $Y_{-1} = 1$ if one reverses the states $S_0 = +1$ and $S_0 = -1$; hence, one could just adopt the previous policy with the two states reversed. In sum, we need only one optimal policy structure, which in general is of the form

$$U_0 = g^{\mathrm{i}}_{a,b}(S_0, \bar{S}_0) = \begin{cases} 1, & \text{if } S_0 = \bar{S}_0, \text{with probability } a, \\ 0, & \text{if } S_0 = \bar{S}_0, \text{with probability } 1 - a, \\ 1, & \text{if } S_0 = -\bar{S}_0, \text{with probability } b, \\ 0, & \text{if } S_0 = -\bar{S}_0, \text{with probability } 1 - b. \end{cases} \tag{31}$$

Define $g^{\mathrm{i}}[j, k]$ to be the policy that A1 plays $U_0 = j$ deterministically when $S_0 = \bar{S}_0$ and $U_0 = k$ deterministically when $S_0 = -\bar{S}_0$, where $j, k \in \{0, 1\}$. Then the policy in (31) can be seen as a "mixed policy" of the four "pure policy" $g^{\mathrm{i}}[0, 0]$, $g^{\mathrm{i}}[0, 1]$, $g^{\mathrm{i}}[1, 0]$, and $g^{\mathrm{i}}[1, 1]$; we know from stochastic control that the value of $g^{\mathrm{i}}_{a,b}$ will be a linear combination of those of $g^{\mathrm{i}}[0, 0]$, $g^{\mathrm{i}}[0, 1]$, $g^{\mathrm{i}}[1, 0]$, and $g^{\mathrm{i}}[1, 1]$. It is clear that one of the 'pure policy" must be optimal. Note that $g^{\mathrm{i}}[1, 1]$ will never be the unique optimal policy. This is because both policies $g^{\mathrm{i}}[0, 1]$ and $g^{\mathrm{i}}[1, 0]$ can transmit one bit of information so that A2 can completely decode S_0 from U_0 and thus S_0 becomes CI. Adopting the policy $g^{\mathrm{i}}[1, 1]$ could potentially increase the communication cost if $c > 0$ and $\epsilon > 0$, but cannot increase the expected reward, since with $g^{\mathrm{i}}[0, 1]$ and $g^{\mathrm{i}}[1, 0]$ they will already be able to getting the matching reward 1 for sure. Although when A1 does communicate ($U_0 = 1$), A2 directly know the state from $Z_0 = U_0 S_0 = S_0$, this feature is not necessary – A2 can simply infer S_0 from U_0 and the model parameters.

It is clear that when $g^{\mathrm{i}}[0, 1]$ or $g^{\mathrm{i}}[1, 0]$ is adopted, S_0 also becomes CI, and A2 simply matches it – not matching S_0 will only lead to a zero instantaneous reward, and this does not increase the "continuation reward" (expected future discounted reward) either, since it is fixed as mentioned previously. On the other hand, when $g^{\mathrm{i}}[0, 0]$ is adopted, A2 can only make a maximum likelihood estimation based on the knowledge of the realizations of $\bar{S}_0$ and O_0; that is, it compares $\mathbb{P}(S_0 = \bar{S}_0 | \bar{S}_0, O_0)$ and $\mathbb{P}(S_0 = -\bar{S}_0 | \bar{S}_0, O_0)$, and matches the S_0 with the higher probability. The expected instantaneous reward of adopting $g^{\text{i}}[0,0]$ is thus $\mathbb{P}(Y_0 = 1)$, which falls into the following cases. If A2 observes $O_0 = \bar{S}_0 = S_{-1} \cdot (-1)^{Y_{-1}+1}$ (the $1-\epsilon$ branch), there is a chance of $(1-\epsilon)\beta$ where its observation is actually correct, and a chance of $\epsilon(1-\beta)$ where it is wrong. Hence, to maximize $\mathbb{P}(Y_0 = 1)$, A2 only follows the observation if $(1-\epsilon)\beta > \epsilon(1-\beta)$, or $\epsilon < \beta$. Similarly, if A2 observes $O_0 = -\bar{S}_0 = S_{-1} \cdot (-1)^{Y_{-1}}$ (the ϵ branch), there is a chance of $\epsilon\beta$ where its observation is actually correct, and a chance of $(1-\epsilon)(1-\beta)$ where it is wrong. Hence, to maximize $\mathbb{P}(Y_0 = 1)$, A2 only follows the observation if $\epsilon\beta > (1-\epsilon)(1-\beta)$, or $\epsilon + \beta > 1$. Depending on whether A2 follows the observation when $O_t = \bar{S}_0 = S_{t-1} \cdot (-1)^{Y_{t-1}+1}$ and whether A2 follows the observation when $O_t = -\bar{S}_0 = S_{t-1} \cdot (-1)^{Y_{t-1}}$, a given model falls into one of the following four cases:

- $\epsilon < \beta$ and $\epsilon + \beta > 1$: A2 always follows the observations, $\mathbb{P}(Y_0 = 0) = \bar{\epsilon}\bar{\beta} + \epsilon\bar{\beta} = \bar{\beta}$;
- $\epsilon < \beta$ and $\epsilon + \beta < 1$: A2 always assumes the $\bar{\epsilon} = 1 - \epsilon$ branch ($S_t = S_{t-1} \cdot (-1)^{Y_{t-1}+1}$), $\mathbb{P}(Y_0 = 0) = \epsilon\bar{\beta} + \epsilon\beta = \epsilon$;
- $\epsilon > \beta$ and $\epsilon + \beta > 1$: A2 always assumes the ϵ branch ($S_t = S_{t-1} \cdot (-1)^{Y_{t-1}}$), $\mathbb{P}(Y_0 = 0) = \bar{\epsilon}\beta + \bar{\epsilon}\bar{\beta} = \bar{\epsilon}$;
- $\epsilon > \beta$ and $\epsilon + \beta < 1$: A2 always flips the observations, $\mathbb{P}(Y_0 = 0) = \bar{\epsilon}\beta + \epsilon\beta = \beta$.

Back to the decision of U_0, A1 compares the instantaneous rewards of the three policies $g^{\text{i}}[0,0]$, $g^{\text{i}}[0,1]$, and $g^{\text{i}}[1,0]$, and chooses the highest. The instantaneous reward of $g^{\text{i}}[0,1]$ is $1 - \epsilon c$ (it communicates ϵ of the time), and the instantaneous reward of $g^{\text{i}}[0,1]$ is $1-(1-\epsilon)c$. Also recall the instantaneous reward of $g^{\text{i}}[0,0]$ is $\mathbb{P}(Y_0 = 1)$ under the policy. Let U_0' be the indicator that one of $g^{\text{i}}[0,1]$ and $g^{\text{i}}[1,0]$ is optimal; that is, $U_0' = 1$ if A1 should convey the information of S_0 through a protocol, and $U_0' = 0$ if A1 should just leave A2 to guess S_0 with maximum likelihood estimation. Then the optimal choice of U_0' is

$$\begin{aligned} U_0' &= \begin{cases} 1, & \text{if } \mathbb{P}(Y_0 = 1) < 1 - \epsilon c \text{ or } \mathbb{P}(Y_0 = 1) < 1 - (1-\epsilon)c, \\ 0, & \text{if } \mathbb{P}(Y_0 = 1) > 1 - \epsilon c \text{ and } \mathbb{P}(Y_0 = 1) > 1 - (1-\epsilon)c. \end{cases} \\ &= \begin{cases} 1, & \text{if } \mathbb{P}(Y_0 = 0) < \min\{\epsilon, 1-\epsilon\}c, \\ 0, & \text{if } \mathbb{P}(Y_0 = 0) > \min\{\epsilon, 1-\epsilon\}c. \end{cases} \end{aligned} \tag{32}$$

Now we change the indexing back to t. Combining (32) with the four cases listed above gives (16), and combining (16) with the definition of $g^{\text{i}}[j,k]$ gives (15).

Appendix C. Proofs of the Regret Bounds of the Two-Stage UCB Algorithm

C.1. The K-arm UCB

In this subsection, we prove one regret bound of the UCB1 algorithm (Auer et al., 2002) that will be used later. At time t, define $\hat{\mu}_t(k) = \frac{1}{n_t(k)} \sum_{s=1}^{t-1} R_s(k_s) \mathbf{1}\{k_s = k\}$ to be the empirical mean of the k-th arm based on it being played $n_t(k) = \sum_{s=1}^{t-1} \mathbf{1}\{k_s = k\}$ times up until $t-1$. As in the two-stage setting, we assume the best arm is the first.

In the following, all the t's and T's within any bound is assumed to be sufficiently large (in particular $> K$), so that we do not have the initialization issue.

Algorithm 2 K-arm UCB

input: K (#actions)
for $t = 1, \dots, K$ **do**
  Play each arm k once, where $k \in [K]$.
end
for $t = K+1, \dots, T$ **do**
  Chooses $k_t = \arg\max_{k \in [K]} \ \hat{\mu}_t(k) + 2\sqrt{\frac{\log(KT/\delta)}{n_t(k)}}$.
end

Lemma 12 *For any $t \in [T] \setminus [K]$ and $k \in [K]$, with probability at least $1-\delta$, we have*

$$|\mu_k - \hat{\mu}_t(k)| \le 2\sqrt{\frac{\log(1/\delta)}{n_t(k)}}.$$

Proof This is Hoeffding's inequality. ■

Corollary 13 *With probability at least $1-\delta$, the inequality*

$$|\mu_k - \hat{\mu}_t(k)| \le 2\sqrt{\frac{\log(KT/\delta)}{n_t(k)}}.$$

holds for all $t \in [T] \setminus [K]$ and $k \in [K]$ simultaneously.

Proof From Lemma 12, the inequality holds for any $t \in [T] \setminus [K]$ and $k \in [K]$ with probability at least $1 - \delta/KT$. Hence, it holds for all $t \in [T] \setminus [K]$ and $k \in [K]$ simultaneously with probability at least $1-\delta$ by the union bound. ■

Lemma 14 *With probability at least $1-2\delta$, the regret is bounded by*

$$\sum_{t=1}^{T} [\mu_1 - R_t(k_t)] \le 11\sqrt{KT\log(KT/\delta)}.$$

Proof With probability at least $1-\delta$, for all $t \in [T] \setminus [K]$ simultaneously,

$$\begin{aligned}
\mu_1 &\le \hat{\mu}_t(1) + 2\sqrt{\frac{\log(KT/\delta)}{n_t(1)}} && \text{(Corollary 13)}\\
&\le \hat{\mu}_t(k_t) + 2\sqrt{\frac{\log(KT/\delta)}{n_t(k_t)}} && \text{(by the way } k_t \text{ is selected)}\\
&\le \mu_{k_t} + 4\sqrt{\frac{\log(KT/\delta)}{n_t(k_t)}}. && \text{(Corollary 13)}
\end{aligned}$$

Therefore, with probability at least $1-\delta$,

$$\begin{aligned}
\sum_{t=1}^{T}(\mu_1-\mu_{k_t}) &\leq K+4\sum_{t=K+1}^{T}\sqrt{\frac{\log(KT/\delta)}{n_t(k_t)}} \\
&= K+4\sum_{k\in[K]}\sum_{t=K+1}^{T}\mathbf{1}\{k_t=k\}\sqrt{\frac{\log(KT/\delta)}{n_t(k_t)}} \\
&= K+4\sum_{k\in[K]}\left(\sqrt{\frac{1}{1}}+\sqrt{\frac{1}{2}}+\cdots+\sqrt{\frac{1}{n_T(k)}}\right)\sqrt{\log(KT/\delta)} \\
&\leq K+8\sum_{k\in[K]}\sqrt{n_T(k)}\cdot\sqrt{\log(KT/\delta)} && \text{(Integral test for series)} \\
&\leq K+8\sqrt{K\cdot\sum_{k\in[K]}n_T(k)}\cdot\sqrt{\log(KT/\delta)} && \text{(Cauchy-Schwartz inequality)} \\
&= 8\sqrt{KT\log(KT/\delta)}+K.
\end{aligned}$$

Notice that $\{\sum_{s=1}^{t}[\mu_{k_s}-R_s(k_s)]\}$ forms a martingale, and $|\mu_{k_t}-R_t(k_t)|\leq 1$. Thus, by Azuma's inequality, with probability at least $1-\delta$,

$$\sum_{t=1}^{T}[\mu_{k_t}-R_t(k_t)]\leq 2\sqrt{T\log(1/\delta)}.$$

Using the union bound, we have with probability at least $1-2\delta$,

$$\begin{aligned}
\sum_{t=1}^{T}[\mu_1-R_t(k_t)] &\leq 8\sqrt{KT\log(KT/\delta)}+2\sqrt{T\log(1/\delta)}+K \\
&\leq 11\sqrt{KT\log(KT/\delta)}.
\end{aligned}$$

■

C.2. Proof of Theorem 9

Lemma 15 *For any $t\in[T]\setminus[KL]$, $k\in[K]$, and $l\in[L]$, with probability at least $1-2\delta$,*

$$\sum_{s=1}^{t}\mathbf{1}\{k_s=k\}\left(\mu_{k,1}-R_s(k,l_s)\right)\leq 11\sqrt{Ln_t^1(k)\log(Ln_t^1(k)/\delta)}\leq 11\sqrt{Ln_t^1(k)\log(LT/\delta)}.$$

Proof By the UCB1 regret bound given in Lemma 14 for the second layer when the k-th arm is selected in the first layer. ■

Lemma 16 *For any $t\in[T]\setminus[KL]$, $k\in[K]$, and $l\in[L]$, with probability at least $1-2\delta$,*

$$\mu_{k,1}\leq\hat{\mu}_t^1(k)+11\sqrt{\frac{L\log(LT/\delta)}{n_t^1(k)}}.$$

Proof Dividing all the terms in Lemma 15 by $n_t^1(k)$ gives the inequality. ■

Proof

$$\sum_{t=1}^{T}[\mu_{1,1} - R_t(k_t, l_t)]$$

$$\leq KL + \sum_{t=KL+1}^{T} \left\{ \max_k \left[\hat{\mu}_t^1(k) + 11\sqrt{\frac{L\log(LT/\delta)}{n_t^1(k)}} \right] - \mu_{k_t,1} + \mu_{k_t,1} - R_t(k_t, l_t) \right\}$$

(Lemma 16 and taking max)

$$= KL + \sum_{t=KL+1}^{T} \left\{ \hat{\mu}_t^1(k_t) + 11\sqrt{\frac{L\log(LT/\delta)}{n_t^1(k_t)}} - \mu_{k_t,1} + \mu_{k_t,1} - R_t(k_t, l_t) \right\}.$$

(by the way k_t is selected)

Notice that for any $t \in [T]$ and $k \in [K]$,

$$\hat{\mu}_t^1(k) - \mu_{k,1} = \frac{1}{n_t^1(k)} \sum_{s=1}^{t-1} \mathbf{1}\{k_s = k\}[R_s(k_s, l_s) - \mu_{k,1}] \leq \frac{1}{n_t^1(k)} \sum_{s=1}^{t-1} \mathbf{1}\{k_s = k\}[R_s(k_s, l_s) - \mu_{k,l_s}].$$

The term $\mathbf{1}\{k_s = k\}[R_s(k_s, l_s) - \mu_{k,l_s}]$ has zero mean, and its absolute value is bounded by $\mathbf{1}\{k_s = k\}$. Hence, by Azuma's inequality, with high probability at least $1 - \delta$,

$$\sum_{s=1}^{t-1} \mathbf{1}\{k_s = k\}[R_s(k_s, l_s) - \mu_{k,l_s}] \leq 2\sqrt{\sum_{s=1}^{t-1} \mathbf{1}\{k_s = k\} \log(1/\delta)} = 2\sqrt{n_t^1(k) \log(1/\delta)}.$$

Combining the two inequalities gives

$$\hat{\mu}_t^1(k_t) - \mu_{k_t,1} \leq 2\sqrt{\frac{\log(1/\delta)}{n_t^1(k_t)}},$$

so that with probability at least $1 - \delta$,

$$\hat{\mu}_t^1(k_t) - \mu_{k_t,1} \leq 2\sqrt{\frac{\log(T/\delta)}{n_t^1(k_t)}},$$

holds for all $t \in [T]$ simultaneously by union bound. So the sum of the first three terms is bounded by

$$\sum_{t=KL+1}^{T} 13\sqrt{\frac{L\log(LT/\delta)}{n_t^1(k_t)}} \leq 26\sqrt{KT} \cdot \sqrt{L\log(LT/\delta)} \leq O(\sqrt{KLT\log(LT/\delta)}).$$

For how to bound $\sum_{t=KL+1}^{T} \sqrt{\frac{1}{n_t^1(k_t)}}$ by $2\sqrt{KT}$, see the proof of Lemma 14. The last two terms sum up to

$$\sum_{t=KL+1}^{T} \mu_{k_t,1} - R_t(k_t, l_t) = \sum_{k \in [K]} \sum_{t=1}^{T} \mathbf{1}\{k_t = k\} \left[\mu_{k,1} - R_t(k, l_t)\right] \quad \text{(Lemma 15)}$$

$$\leq \sum_{k\in[K]} O\left(\sqrt{Ln_T^1(k)\log(LT/\delta)}\right) \quad \text{(Cauchy-Schwartz inequality)}$$
$$= O\left(\sqrt{KLT\log(LT/\delta)}\right).$$

Combining all parts, with probability at least $1-5\delta$, or simply $1-\delta$ as the constant term can be absorbed into the big-O notation,

$$\sum_{t=1}^{T}[\mu_{1,1} - R_t(k_t, l_t)] \leq O\left(\sqrt{KLT\log(LT/\delta)}\right).$$

■

C.3. Proof of Theorem 10

Lemma 17 *Let $k \neq 1$ be a sub-optimal arm of A1. Then with probability at least $1-2\delta$,*

$$\sum_{t=1}^{T}\mathbf{1}\{k_t = k\} \leq O\left(\frac{L\log(LT/\delta)}{(\mu_{1,1}-\mu_{k,1})^2}\right).$$

Proof Suppose k has been drawn for $\frac{26^2 L\log(LT/\delta)}{(\mu_{1,1}-\mu_{k,1})^2}$ times before time t. That is, $n_t^1(k) \geq \frac{26^2 L\log(LT/\delta)}{(\mu_{1,1}-\mu_{k,1})^2}$. Then with probability at least $1-\delta$,

$$\hat{\mu}_t^1(k) + 11\sqrt{\frac{L\log(LT/\delta)}{n_t^1(k)}}$$
$$\leq \mu_{k,l_t} + 2\sqrt{\frac{\log(LT/\delta)}{n_t^1(k)}} + 11\sqrt{\frac{L\log(LT/\delta)}{n_t^1(k)}} \quad \text{(Corollary 13)}$$
$$\leq \mu_{k,1} + 13\sqrt{\frac{L\log(LT/\delta)}{n_t^1(k)}} \quad \text{(First arm is optimal)}$$
$$\leq \mu_{k,1} + (\mu_{1,1}-\mu_{k,1})/2 < \mu_{1,1}.$$

But by Lemma 16, with probability at least $1-\delta$,

$$\mu_{1,1} \leq \hat{\mu}_t^1(1) + 11\sqrt{\frac{L\log(LT/\delta)}{n_t^1(1)}}.$$

By the algorithm's way of choosing k_t, A1 will not choose arm k for all $t \in [T]$ with probability at least $1-2\delta$, as long as the condition of $n_t^1(k)$ is satisfied. ■

Lemma 18 *Let $l \neq 1$ be a sub-optimal arm of A2 given A1 chooses k. Then with probability at least $1-2\delta$,*

$$\sum_{t=1}^{T}\mathbf{1}\{k_t = k, l_t = l\} \leq O\left(\frac{\log(LT/\delta)}{(\mu_{k,1}-\mu_{k,l})^2}\right).$$

Proof Suppose at time t, $n_t^2(k,l) \geq \frac{64\log(LT/\delta)}{(\mu_{k,1}-\mu_{k,l})^2}$. Then with probability at least $1-\delta$,

$$\begin{aligned}
&\hat{\mu}_t^2(k,l) + 2\sqrt{\frac{\log(LT/\delta)}{n_t^2(k,l)}} \\
&\leq \mu_{k,l} + 4\sqrt{\frac{\log(LT/\delta)}{n_t^2(k,l)}} && \text{(Corollary 13)} \\
&\leq \mu_{k,l} + \frac{(\mu_{k,1}-\mu_{k,l})}{2} < \mu_{k,1}.
\end{aligned}$$

But by Corollary 13, with probability at least $1-\delta$,

$$\mu_{k,1} \leq \hat{\mu}_t^2(k,1) + 2\sqrt{\frac{\log(LT/\delta)}{n_t^2(k,1)}}.$$

So for all the round $t \in [T]$, even if A1 chooses k, with probability at least $1-2\delta$, A2 will not choose l given the condition of $n_t^2(k,l)$ is satisfied. ■

Proof We show that the regret is upper bounded by

$$\sum_{(k,l)\neq(1,1)} (\mu_{1,1}-\mu_{k,l}) \cdot n_{T+1}^2(k,l) \leq L\sum_{k\neq 1} O\left(\frac{\log(LT)}{\mu_{1,1}-\mu_{k,1}}\right) + \sum_k \sum_{l\neq 1} O\left(\frac{\log(LT)}{\mu_{k,1}-\mu_{k,l}}\right).$$

Lemma 17 gives with probability at least $1-2\delta$,

$$\sum_l n_{T+1}^2(k,l) \leq O\left(\frac{L\log(LT/\delta)}{(\mu_{1,1}-\mu_{k,1})^2}\right) \quad \forall k.$$

Lemma 18 gives with probability at least $1-2\delta$,

$$n_{T+1}^2(k,l) \leq O\left(\frac{\log(LT/\delta)}{(\mu_{k,1}-\mu_{k,l})^2}\right).$$

Thus, with probability at least $1-4\delta$, we can bound the regret by

$$\begin{aligned}
&\sum_{(k,l)\in[K]\times[L]\backslash(1,1)} (\mu_{1,1}-\mu_{k,l}) \cdot n_{T+1}^2(k,l) \\
&= \sum_{(k,l)\in[K]\times[L]\backslash(1,1)} (\mu_{1,1}-\mu_{k,1}+\mu_{k,1}-\mu_{k,l}) \cdot n_{T+1}^2(k,l) \\
&= \sum_{k\in[K]\backslash\{1\}} (\mu_{1,1}-\mu_{k,1}) \cdot \sum_{l\in[L]} n_{T+1}^2(k,l) + \sum_{k\in[K]}\sum_{l\in[L]\backslash\{1\}} (\mu_{k,1}-\mu_{k,l}) \cdot n_{T+1}^2(k,l) \\
&\leq \sum_{k\in[K]\backslash\{1\}} (\mu_{1,1}-\mu_{k,1}) \cdot O\left(\frac{L\log(LT/\delta)}{(\mu_{1,1}-\mu_{k,1})^2}\right) + \sum_{k\in[K]}\sum_{l\in[L]\backslash\{1\}} (\mu_{k,1}-\mu_{k,l}) \cdot O\left(\frac{\log(LT/\delta)}{(\mu_{k,1}-\mu_{k,l})^2}\right) \\
&= L\sum_{k\in[K]\backslash\{1\}} O\left(\frac{\log(LT/\delta)}{\mu_{1,1}-\mu_{k,1}}\right) + \sum_{k\in[K]}\sum_{l\in[L]\backslash\{1\}} O\left(\frac{\log(LT/\delta)}{\mu_{k,1}-\mu_{k,l}}\right).
\end{aligned}$$

The constant term of 4 can be easily absorbed into the big-O notation. ■

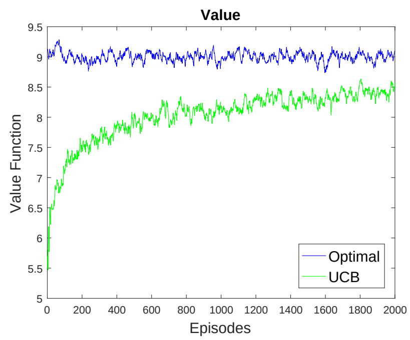

Figure 2: Value function with episodes (no communication).

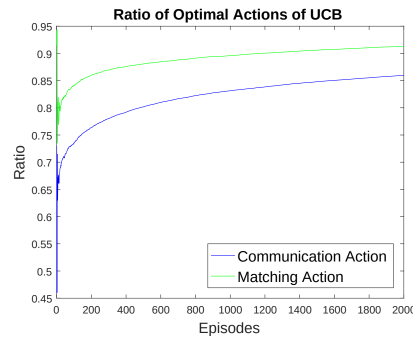

Figure 3: Ratio of optimal actions with episodes (no communication).

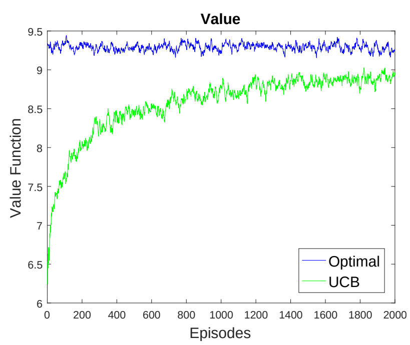

Figure 4: Value function with episodes (with communication).

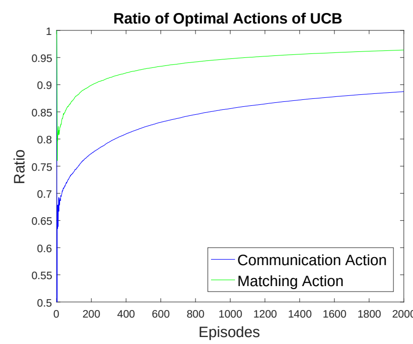

Figure 5: Ratio of optimal actions with episodes (with communication).

Appendix D. Simulation Results of the Two-Stage UCB Algorithm and the Optimal Policy

The following is a set of parameters that lead to no communication in the optimal policy (16): $\epsilon = 0.2$, $\beta = 0.9$, $\lambda = 0.9$, $c = 0.9$, and $T = 2500$. In A2's optimal policy (18), it follows the observation when $O_t = S_{t-1} \cdot (-1)^{Y_{t-1}+1}$, but flips the observation when $O_t = S_{t-1} \cdot (-1)^{Y_{t-1}}$. So basically A2 just assumes $S_t = S_{t-1} \cdot (-1)^{Y_{t-1}+1}$ is always the case, which lead to an error probability of ϵ. The expected reward is thus $1 - \epsilon = 0.9$, and the value function at any time step is $\frac{1-\epsilon}{1-\lambda} = \frac{0.9}{1-0.9} = 9$.

Figure 2 shows the value function, which is calculated by $V(t) \approx \sum_{i=t}^{T} R(t)\lambda^{i-t}$ (instead of summing to infinity). The result is averaged from 100 runs. The optimal policy (known model) just stays at 9, while two-stage UCB starts from around 5 and approaches 9 as $t \to \infty$. Figure 3 depicts the ratio of their optimal actions, namely, the portion that A1 chooses to not communicate out of all of its choices, and the portion that A2 just assumes $S_t = S_{t-1} \cdot (-1)^{Y_{t-1}+1}$ out of all of its choices. They do approach 1 as $t \to \infty$.

Another set of parameters where A1 finds it beneficial to communicate to A2 through the protocol given in (15) and (17): $\epsilon = 0.1$, $\beta = 0.7$, $\lambda = 0.9$, $c = 0.7$, and $T = 2500$. Figure 4 shows the value function, and Figure 5 depicts the ratio of their optimal actions. The value function at any time step is given by $\frac{1-\min\{\epsilon,\bar{\epsilon}\}c}{1-\lambda} = \frac{1-0.1\cdot 0.7}{1-0.9} = 9.3$.